2018 AIME II Solutions

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1. Points $A, B,$ and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C, Paul starting at B and running toward C, and Eve starting at C and running toward A. When Paul meets Eve, he turns around and runs toward A. Paul and Ina both arrive at B at the same time. Find the number of meters from A to B.

Solution:

Let $x = AB$, so $BC = 1800 - x$, and let Eve's speed be v, so Ina runs at $2v$ and Paul at $4v$. Paul and Eve start at B and C running toward each other, so together they cover the $1800 - x$ meters between them, with Paul covering $\frac{4}{5}$ of it. Paul then retraces that distance back to B, so when he reaches B he has run $\frac{8}{5}(1800 - x)$ meters in total.

Ina reaches B at the same moment, having run x meters. Since Paul runs twice as fast as Ina, he has run $2x$ meters in that time. Therefore

$$\frac{8}{5}(1800 - x) = 2x,$$

which gives $8 \cdot 1800 = 18x$, so $x = 800$.

2. Let $a_0 = 2, a_1 = 5,$ and $a_2 = 8,$ and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by $11.$ Find $a_{2018} \cdot a_{2020} \cdot a_{2022}.$

Solution:

Computing successive terms gives

$$2,\ 5,\ 8,\ 5,\ 6,\ 10,\ 7,\ 4,\ 7,\ 6,\ 2,\ 5,\ 8,\ \ldots$$

Since $(a_{10}, a_{11}, a_{12}) = (2, 5, 8) = (a_0, a_1, a_2)$ and each term depends only on the previous three, the sequence is periodic with period $10.$

Therefore $a_{2018} = a_8 = 7, a_{2020} = a_0 = 2,$ and $a_{2022} = a_2 = 8,$ so the product is $7 \cdot 2 \cdot 8 = 112.$

3. Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

Solution:

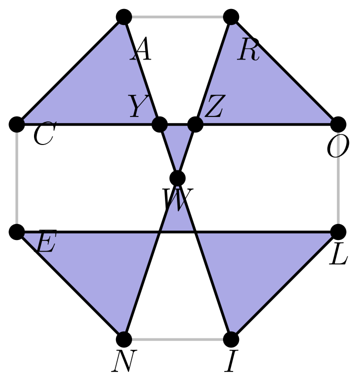
The conditions say $3b + 6$ is a perfect square and $2b + 7$ is a perfect cube. Since $2b + 7$ is odd and $b < 1000$ forces $2b + 7 < 2007,$ the cube must be one of $1, 27, 125, 343, 729, 1331,$ giving $b = -3, 10, 59, 168, 361, 662.$

The corresponding values of $3b + 6$ for the positive candidates are $36, 183, 510, 1089, 1992,$ and only $36 = 6^2$ (for $b = 10$) and $1089 = 33^2$ (for $b = 361$) are perfect squares. The requested sum is $10 + 361 = 371.$

4. In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. Then $K = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Solution:

Since the interior angles are all $135°$ and the $\sqrt{2}$ sides are diagonals of unit squares, the octagon fits on a lattice: $C = (0,0), A = (1,1), R = (2,1), O = (3,0), L = (3,-1), I = (2,-2), N = (1,-2), E = (0,-1)$. The path $CORNELIA$ is carried to itself by the $180°$ rotation about $(\frac{3}{2}, -\frac{1}{2})$. Let Y and Z be the points where AI and RN cross CO, and let $W = AI \cap RN$. Segment AI has slope -3, so $Y = (\frac{4}{3}, 0)$, and by symmetry $Z = (\frac{5}{3}, 0)$ and $W = (\frac{3}{2}, -\frac{1}{2})$.



The six enclosed regions are the four congruent corner triangles like CAY and the two small congruent triangles like YZW. Triangle CAY has base $CY = \frac{4}{3}$ and height 1, so its area is $\frac{2}{3}$. Triangle YZW has base $YZ = \frac{1}{3}$ and height $\frac{1}{2}$, so its area is $\frac{1}{12}$. Therefore

$$K = 4 \cdot \frac{2}{3} + 2 \cdot \frac{1}{12} = \frac{8}{3} + \frac{1}{6} = \frac{17}{6},$$

and $a + b = 17 + 6 = 23$.

5. Suppose that x, y, and z are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96 + 24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x + y + z = a + bi$. Find $a^2 + b^2$.

Solution:

Multiplying the three equations gives

$$(xyz)^2 = (-80 - 320i)(60)(-96 + 24i) = 80 \cdot 60 \cdot 24\,(-1 - 4i)(-4 + i) = 240^2(16 + 30i).$$

Since $16 + 30i = (5 + 3i)^2$, we get $xyz = \pm 240(5 + 3i)$.

Dividing xyz by each given product yields

$$x = \frac{xyz}{yz} = \pm(20 + 12i), \qquad y = \frac{xyz}{zx} = \pm(-10 - 10i), \qquad z = \frac{xyz}{xy} = \pm(-3 + 3i),$$

with matching signs. Hence $x + y + z = \pm(7 + 5i)$, so $(a, b) = \pm(7, 5)$ and $a^2 + b^2 = 49 + 25 = 74$.

6. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a - 2)x^2 + (-4a + 3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Group the terms by whether they involve a :

$$(x^4 - 2x^2 + 3x - 2) + 2a(x^3 + x^2 - 2x) = (x - 1)(x + 2)(x^2 - x + 1) + 2ax(x - 1)(x + 2),$$

so the polynomial factors as $(x - 1)(x + 2)\left(x^2 + (2a - 1)x + 1\right)$.

All four roots are real exactly when the quadratic factor has real roots, i.e. when $(2a - 1)^2 - 4 \geq 0$, which means $a \leq -\frac{1}{2}$ or $a \geq \frac{3}{2}$. The excluded interval $\left(-\frac{1}{2}, \frac{3}{2}\right)$ has length 2 inside $[-20, 18]$, which has length 38, so the probability is $\frac{36}{38} = \frac{18}{19}$. The requested sum is $18 + 19 = 37$.

7. Triangle ABC has side lengths $AB = 9, BC = 5\sqrt{3}$, and $AC = 12$. Points $A = P_0, P_1, P_2, \ldots, P_{2450} = B$ are on segment $\overline{AB}$ with P_k between P_{k-1} and P_{k+1} for $k = 1, 2, \ldots, 2449$, and points $A = Q_0, Q_1, Q_2, \ldots, Q_{2450} = C$ are on segment $\overline{AC}$ with Q_k between Q_{k-1} and Q_{k+1} for $k = 1, 2, \ldots, 2449$. Furthermore, each segment $\overline{P_kQ_k}, k = 1, 2, \ldots, 2449$, is parallel to $\overline{BC}$. The segments cut the triangle into 2450 regions, consisting of 2449 trapezoids and 1 triangle. Each of the 2450 regions has the same area. Find the number of segments $\overline{P_kQ_k}, k = 1, 2, \ldots, 2450$, that have rational length.

Solution:

Since the 2450 regions have equal areas, triangle AP_kQ_k (the union of the first k regions) has area $\frac{k}{2450}$ of triangle ABC. Each triangle AP_kQ_k is similar to ABC, and lengths scale as the square root of areas, so

$$P_kQ_k = 5\sqrt{3}\sqrt{\frac{k}{2450}} = 5\sqrt{3} \cdot \frac{\sqrt{k}}{35\sqrt{2}} = \frac{\sqrt{6k}}{14}.$$

This is rational exactly when $6k$ is a perfect square, which happens exactly when $k = 6j^2$ for a positive integer j. The condition $6j^2 \leq 2450$ gives $j^2 \leq 408$, so $j = 1, 2, \ldots, 20$. There are 20 such segments.

8. A frog is positioned at the origin in the coordinate plane. From the point (x, y), the frog can jump to any of the points $(x+1, y)$, $(x+2, y)$, $(x, y+1)$, or $(x, y+2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0, 0)$ and ends at $(4, 4)$.

Solution:

The horizontal jumps are steps of 1 or 2 summing to 4, so as a multiset they are $\{1,1,1,1\}$, $\{1,1,2\}$, or $\{2,2\}$, and the same holds for the vertical jumps. For any choice of the two multisets, every ordering of all the jumps is a valid sequence, and the number of orderings is the multinomial coefficient of the combined multiset.

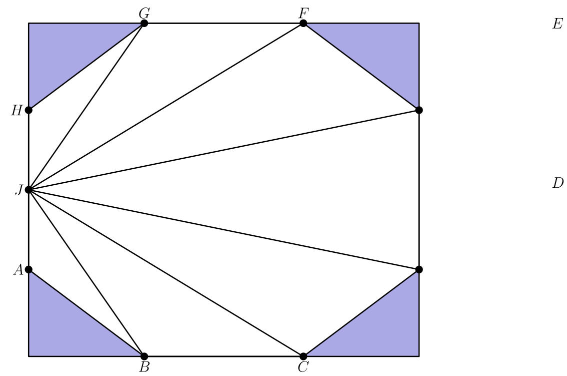
The nine cases give

$$\binom{8}{4} = 70, \quad \frac{7!}{4!\,2!} = 105 \text{ (twice)}, \quad \frac{6!}{4!\,2!} = 15 \text{ (twice)},$$

$$\frac{6!}{2!\,2!} = 180, \quad \frac{5!}{2!\,2!} = 30 \text{ (twice)}, \quad \binom{4}{2} = 6.$$

The total is $70 + 2 \cdot 105 + 2 \cdot 15 + 180 + 2 \cdot 30 + 6 = 556$.

9. Octagon $ABCDEFGH$ with side lengths $AB = CD = EF = GH = 10$ and $BC = DE = FG = HA = 11$ is formed by removing four 6-8-10 triangles from the corners of a 23×27 rectangle with side $\overline{AH}$ on a short side of the rectangle, as shown. Let J be the midpoint of $\overline{AH}$, and partition the octagon into 7 triangles by drawing segments $\overline{JB}, \overline{JC}, \overline{JD}, \overline{JE}, \overline{JF}$, and $\overline{JG}$. Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.



Solution:

Each of the 7 triangles has J as a vertex, and the centroid of a triangle JVW lies on the segment from J to the midpoint of $\overline{VW}$, two-thirds of the way out. So the centroid heptagon is the image of the heptagon S formed by the midpoints of $\overline{AB}, \overline{BC}, \ldots, \overline{GH}$ under a dilation centered at J with ratio $\frac{2}{3}$, and its area is $\frac{4}{9}[S]$.

Place the rectangle with $A = (0,6), B = (8,0), C = (19,0), D = (27,6), E = (27,17), F = (19,23), G = (8,23), H = (0,17)$, so $J = (0, \frac{23}{2})$. The midpoints are $(4,3), (\frac{27}{2}, 0), (23,3), (27, \frac{23}{2}), (23,20), (\frac{27}{2}, 23), (4,20)$. The vertical segments at $x = 4, x = \frac{27}{2}$, and $x = 23$ have lengths $17, 23$, and 17, cutting S into two trapezoids of height $\frac{19}{2}$ and a triangle of height 4 :

$$[S] = 2 \cdot \frac{17+23}{2} \cdot \frac{19}{2} + \frac{17 \cdot 4}{2} = 380 + 34 = 414.$$

The requested area is $\frac{4}{9} \cdot 414 = 184.$

10. Find the number of functions $f(x)$ from $\{1,2,3,4,5\}$ to $\{1,2,3,4,5\}$ that satisfy $f(f(x)) = f(f(f(x)))$ for all x in $\{1,2,3,4,5\}$.

Solution:

Applying f to $f(f(x)) = f(f(f(x)))$ repeatedly shows the condition means that $f(f(x))$ is a fixed point of f for every x. So the elements organize into levels: a nonempty set of i fixed points, then j elements whose image is a fixed point (but which are not fixed), and the remaining $5-i-j$ elements, each of which must map to one of the j middle elements.

For given i and j there are $\binom{5}{i}$ choices of fixed points, $\binom{5-i}{j}$ choices of the middle level, i^j maps from the middle level to the fixed points, and j^{5-i-j} maps for the rest. Summing $\binom{5}{i}\binom{5-i}{j}\, i^j\, j^{5-i-j}$ over the valid pairs (all $i \geq 1, j \geq 1$, plus the identity case $i = 5$) gives

$$20 + 120 + 60 + 5 + 60 + 240 + 80 + 60 + 90 + 20 + 1 = 756.$$

11. Find the number of permutations of $1,2,3,4,5,6$ such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k.

Solution:

The condition fails exactly when the first k terms are a permutation of $\{1,\ldots,k\}$ for some $k \leq 5$. For a permutation of $1,\ldots,n$, let k be the smallest length for which the prefix is $\{1,\ldots,k\}$ (the full length n always works), and let c_n be the number of permutations whose smallest such k is n. We want c_6.

Every permutation of $1,\ldots,n$ decomposes uniquely as a minimal prefix of length k (c_k choices) followed by any arrangement of the remaining $n-k$ values, so

$$\sum_{k=1}^{n} c_k\,(n-k)! = n!.$$

Starting from $c_1 = 1$, this gives $c_2 = 1, c_3 = 3, c_4 = 13, c_5 = 71$, and

$$c_6 = 720 - (120\cdot 1 + 24\cdot 1 + 6\cdot 3 + 2\cdot 13 + 1\cdot 71) = 720 - 259 = 461.$$

12. Let $ABCD$ be a convex quadrilateral with $AB = CD = 10, BC = 14,$ and $AD = 2\sqrt{65}.$ Assume that the diagonals of $ABCD$ intersect at point $P,$ and that the sum of the areas of triangles APB and CPD equals the sum of the areas of triangles BPC and $APD.$ Find the area of quadrilateral $ABCD.$

Solution:

Let $a = AP, b = BP, c = CP, d = DP,$ and let $\theta = \angle CPD.$ Since $\sin(\pi - \theta) = \sin\theta,$ the equal-area condition $\frac{1}{2}(ab + cd)\sin\theta = \frac{1}{2}(ad + bc)\sin\theta$ simplifies to $(a - c)(d - b) = 0.$ By symmetry assume $a = c.$

The law of cosines in triangles BPC and APB (whose angles at P are supplementary) gives $a^2 + b^2 + 2ab\cos\theta = 196$ and $a^2 + b^2 - 2ab\cos\theta = 100,$ so $a^2 + b^2 = 148$ and $ab\cos\theta = 24.$ Similarly triangles APD and CPD give $a^2 + d^2 = 180$ and $ad\cos\theta = 40.$ Dividing, $\frac{d}{b} = \frac{5}{3},$ while subtracting gives $d^2 - b^2 = 32;$ hence $b = 3\sqrt{2}, d = 5\sqrt{2}, a^2 = 130,$ and $\cos^2\theta = \frac{24^2}{130\cdot 18} = \frac{16}{65},$ so $\sin\theta = \frac{7}{\sqrt{65}}.$

The total area is

$$\frac{1}{2}(a + c)(b + d)\sin\theta = a(b + d)\sin\theta = \sqrt{130}\cdot 8\sqrt{2}\cdot\frac{7}{\sqrt{65}} = 112.$$

13. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Let a be the probability that the total number of rolls is odd; let b be that probability given that the first roll is a 1, and c given that the first two rolls are 1-2 (in each case counting all rolls). Condition on the next roll, noting that whenever the count restarts, the rolls already used flip the required parity. Starting fresh: a 1 leads to state b; anything else uses one roll, after which an even continuation is needed. After a $1:$ another 1 means the first roll is wasted, needing an even continuation of the b-type; a 2 leads to c; anything else wastes both rolls. After 1-$2:$ a 3 finishes in 3 rolls (odd); a 1 restarts at the b-state with two wasted rolls; anything else wastes all three. Thus

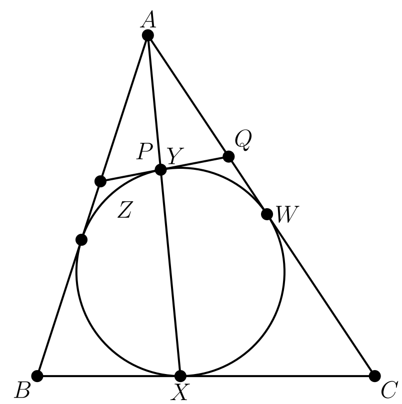
$$a=\frac{1}{6}b+\frac{5}{6}(1-a), \qquad b=\frac{1}{6}(1-b)+\frac{1}{6}c+\frac{4}{6}a, \qquad c=\frac{1}{6}b+\frac{1}{6}+\frac{4}{6}(1-a).$$

The first equation gives $b=11a-5$; substituting the third into the second yields $41b=11+20a$, so $41(11a-5)=11+20a$, giving $431a=216$ and $a=\frac{216}{431}$. Since 431 is prime, $m+n=216+431=647$.

14. The incircle ω of triangle ABC is tangent to $\overline{BC}$ at X. Let $Y \neq X$ be the other intersection of $\overline{AX}$ with ω. Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y. Assume that $AP = 3, PB = 4, AC = 8$, and $AQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Let ω touch $\overline{AB}$ at Z and $\overline{AC}$ at W, and set $\alpha = \angle BAX$ and $\beta = \angle AXC$. The tangent-chord angle between PQ and chord XY equals the one between BC and XY, so $\angle QYX = \angle YXC = \beta$, and vertical angles give $\angle AYP = \beta$. In triangle APY the law of sines gives $PY = AP\,\frac{\sin\alpha}{\sin\beta}$, and by equal tangents $PZ = PY$, so $\frac{AZ}{AP} = 1 + \frac{PY}{AP} = 1 + \frac{\sin\alpha}{\sin\beta}$. In triangle ABX, since $\angle AXB = 180^\circ - \beta$, similarly $BX = AB\,\frac{\sin\alpha}{\sin\beta}$, and $BZ = BX$ gives $\frac{AZ}{AB} = 1 - \frac{\sin\alpha}{\sin\beta}$.



Adding the two relations, $\frac{AZ}{AP} + \frac{AZ}{AB} = 2$, so with $AP = 3$ and $AB = 7$ we get $AZ\left(\frac{1}{3} + \frac{1}{7}\right) = 2$, hence $AZ = \frac{21}{5}$. The identical argument on side AC (using $\angle XAC$ in triangles AQY and ACX) gives $\frac{AW}{AQ} + \frac{AW}{AC} = 2$, and $AW = AZ = \frac{21}{5}$ by equal tangents from A. Therefore

$$\frac{1}{AQ} = \frac{10}{21} - \frac{1}{8} = \frac{59}{168},$$

so $AQ = \frac{168}{59}$ and $m + n = 168 + 59 = 227$.

15. Find the number of functions f from $\{0,1,2,3,4,5,6\}$ to the integers such that $f(0)=0$, $f(6)=12$, and

$$|x-y| \leq |f(x)-f(y)| \leq 3|x-y|$$

for all x and y in $\{0,1,2,3,4,5,6\}$.

Solution:

Let $d_i = f(i) - f(i-1)$, so each $|d_i| \in \{1,2,3\}$ and $d_1 + \cdots + d_6 = 12$. If k of the differences were negative, the sum would be at most $3(6-k) - k = 18 - 4k$, so $k \leq 1$. If no difference is negative, the solutions of $a+b+c=6, a+2b+3c=12$ (with a,b,c counting 1s, 2s, 3s) are $(0,6,0),(1,4,1),(2,2,2),(3,0,3)$, and all such orderings satisfy every pair condition, giving

$$\binom{6}{0,6,0} + \binom{6}{1,4,1} + \binom{6}{2,2,2} + \binom{6}{3,0,3} = 1 + 30 + 90 + 20 = 141.$$

If $d_1 < 0$, then $|f(2)| \geq 2$ with $f(2) \leq d_1 + 3$ forces $d_1 = -1$ and $d_2 = 3$; the remaining four differences are positive and sum to 10, giving $\binom{4}{1,0,3} + \binom{4}{0,2,2} = 4 + 6 = 10$ functions, and the case $d_6 < 0$ is symmetric: 20 in all. Finally, if $d_{n+1} < 0$ for some $n = 1,2,3,4,$ the pair conditions $|f(n+1) - f(n-1)| \geq 2$ and $|f(n+2) - f(n)| \geq 2$ force $d_{n+1} = -1$ and $d_n = d_{n+2} = 3.$ The other three differences are positive and sum to 7, achievable as two 3s and a 1 or a 3 and two 2 s, each in 3 orders: 6 ways for each of the 4 positions, or 24 functions.

The total is $141 + 20 + 24 = 185.$

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