2018 AIME I Solutions

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1. Let S be the number of ordered pairs of integers (a, b), with $1 \leq a \leq 100$ and $b \geq 0$, such that the polynomial $x^2 + ax + b$ can be factored into the product of two (not necessarily distinct) linear factors with integer coefficients. Find the remainder when S is divided by 1000.

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Solution:

The polynomial factors into integer linear factors exactly when its roots are integers, that is, when the discriminant $a^2 - 4b$ equals c^2 for some integer $c \geq 0$. Given a, such a $b \geq 0$ exists exactly when $4b = a^2 - c^2 = (a-c)(a+c)$ for some c with $0 \leq c \leq a$ and $c \equiv a \pmod{2}$, and distinct such c give distinct values $b = \frac{a^2-c^2}{4}$.

For odd a the valid c are $1, 3, \ldots, a$, which is $\frac{a+1}{2}$ choices; for even a they are $0, 2, \ldots, a$, which is $\frac{a}{2} + 1$ choices.

Summing over $a = 1, \ldots, 100:$ the odd a contribute $1 + 2 + \cdots + 50 = 1275$, and the even a contribute $2 + 3 + \cdots + 51 = 1325$. Thus $S = 2600$, and the remainder is 600.

2. The number n can be written in base 14 as $\underline{a}\,\underline{b}\,\underline{c}$, can be written in base 15 as $\underline{a}\,\underline{c}\,\underline{b}$, and can be written in base 6 as $\underline{a}\,\underline{c}\,\underline{a}\,\underline{c}$, where $a > 0$. Find the base-10 representation of n.

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Solution:

Writing out the place values, $n = 196a + 14b + c = 225a + 15c + b = 222a + 37c$, where a and c are base-6 digits with $1 \le a \le 5$ and $0 \le c \le 5$, and $0 \le b \le 13$.

Equating the last two expressions gives $b = 22c - 3a$. Substituting into $196a + 14b + c = 225a + 15c + b$ (which says $13b = 29a + 14c$) yields $13(22c - 3a) = 29a + 14c$, so $272c = 68a$, that is $a = 4c$. The digit bounds force $c = 1, a = 4$, and then $b = 22 - 12 = 10$, which is a valid digit in bases 14 and 15.

Therefore $n = 222 \cdot 4 + 37 \cdot 1 = 925$. Indeed $925 = 196 \cdot 4 + 14 \cdot 10 + 1$, confirming the base-$14$ form. The answer is 925.

3. Kathy has 5 red cards and 5 green cards. She shuffles the 10 cards and lays out 5 of the cards in a row in a random order. She will be happy if and only if all the red cards laid out are adjacent and all the green cards laid out are adjacent. For example, card orders RRGGG, GGGGR, or RRRRR will make Kathy happy, but RRRGR will not. The probability that Kathy will be happy is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

There are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30240$ equally likely ordered layouts of 5 of the 10 distinct cards. Kathy is happy exactly when the color pattern consists of one block of reds and one block of greens: the patterns are RRRRR, GGGGG, and the eight mixed patterns $\mathrm{R}^r\mathrm{G}^{5-r}$ and $\mathrm{G}^{5-r}\mathrm{R}^r$ for $r = 1, 2, 3, 4$.

A pattern using r red and $5-r$ green positions can be filled in $\frac{5!}{(5-r)!} \cdot \frac{5!}{r!}$ ways (ordered choices of which red cards and which green cards appear). For $r = 5, 4, 3, 2, 1, 0$ these counts are $120, 600, 1200, 1200, 600, 120$. The happy layouts number

$$120 + 120 + 2\,(600 + 1200 + 1200 + 600) = 7440.$$

The probability is $\frac{7440}{30240} = \frac{31}{126}$, so $m + n = 31 + 126 = 157$.

4. In $\triangle ABC$, $AB = AC = 10$ and $BC = 12$. Point D lies strictly between A and B on $\overline{AB}$ and point E lies strictly between A and C on $\overline{AC}$ so that $AD = DE = EC$. Then AD can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

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Solution:

By the law of cosines in $\triangle ABC$,

$$\cos A = \frac{10^2 + 10^2 - 12^2}{2 \cdot 10 \cdot 10} = \frac{56}{200} = \frac{7}{25}.$$

Let $x = AD = DE = EC$, so $AE = 10 - x$. The law of cosines in $\triangle ADE$ gives

$$x^2 = x^2 + (10 - x)^2 - 2x(10 - x) \cdot \frac{7}{25},$$

so $(10 - x)^2 = \frac{14}{25}x(10 - x)$. Since $x < 10$, we may divide by $10 - x$ to get $10 - x = \frac{14x}{25}$, hence $250 = 39x$ and $x = \frac{250}{39}$.

As $\gcd(250, 39) = 1$, the answer is $250 + 39 = 289$.

5. For each ordered pair of real numbers (x, y) satisfying

$$\log_2(2x+y) = \log_4(x^2+xy+7y^2),$$

there is a real number K such that

$$\log_3(3x+y) = \log_9(3x^2+4xy+Ky^2).$$

Find the product of all possible values of K.

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Solution:

Because $\log_4 u = \log_2 \sqrt{u}$, the first equation is equivalent to $(2x+y)^2 = x^2 + xy + 7y^2$ together with $2x+y>0$. Expanding gives $3x^2+3xy-6y^2=0$, which factors as $3(x-y)(x+2y)=0$. So $x=y$ or $x=-2y$, with $(x,y) \neq (0,0)$.

Similarly the second equation says $(3x+y)^2 = 3x^2+4xy+Ky^2$, that is $6x^2 + 2xy + y^2 = Ky^2$. If $x=y$ (taking $x>0$ so both logarithms are defined), then $K = 6+2+1 = 9$. If $x=-2y$ (taking $y<0$, so $2x+y=-3y>0$ and $3x+y = -5y > 0$), then $24y^2 - 4y^2 + y^2 = Ky^2$, so $K=21$.

Both cases occur, so the product of all possible values is $9 \cdot 21 = 189$.

6. Let N be the number of complex numbers z with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by $1000.$

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Solution:

Write $z = e^{i\theta}$ with $\theta \in [0, 2\pi)$. Then $z^{720} - z^{120}$ is real exactly when $\sin 720\theta = \sin 120\theta$, which happens when the angles are equal or supplementary modulo 2π : either $720\theta = 120\theta + 2\pi k$, giving $\theta = \frac{\pi k}{300}$, or $720\theta = \pi - 120\theta + 2\pi k$, giving $\theta = \frac{(2k+1)\pi}{840}$.

The first family has 600 values in $[0, 2\pi)$ and the second has $840.$ They cannot coincide: $\frac{\pi k}{300} = \frac{(2j+1)\pi}{840}$ would give $14k = 5(2j+1)$, equating an even number with an odd one.

Hence $N = 600 + 840 = 1440$, and the remainder is $440.$

7. A right hexagonal prism has height 2. The bases are regular hexagons with side length 1. Any 3 of the 12 vertices determine a triangle. Find the number of these triangles that are isosceles (including equilateral triangles).

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Solution:

The chords of a unit regular hexagon have lengths $1, \sqrt{3}$, and 2. Among the $\binom{6}{3} = 20$ triangles in one hexagon, 6 have sides $1, 1, \sqrt{3}$ and 2 are equilateral with side $\sqrt{3}$; the other 12, with sides $1, \sqrt{3}, 2,$ are scalene. So each base contributes 8 isosceles triangles, for 16 in all.

Otherwise two vertices lie on one base (2 choices of that base) and one on the other. A vertex of the top base at horizontal distance d from a bottom vertex is at distance $\sqrt{d^2+4} \geq 2$ from it. If the bottom pair is adjacent (chord 1): the perpendicular bisector of a hexagon edge passes through no vertices, and no slant side can equal 1, so there are no isosceles triangles. If the pair has one vertex between them (chord $\sqrt{3}$, 6 pairs): the top vertices above that middle vertex and above the opposite vertex are equidistant from the pair, giving $6 \cdot 2 = 12$. If the pair is diametrically opposite (chord $2, 3$ pairs): no vertex lies above the perpendicular bisector, but the top vertex directly above either endpoint gives a slant side $\sqrt{0+4} = 2$ equal to the chord, giving $3 \cdot 2 = 6$.

The total is $16 + 2\,(12 + 6) = 52$.

8. Let $ABCDEF$ be an equiangular hexagon such that $AB = 6, BC = 8, CD = 10,$ and $DE = 12.$ Denote by d the diameter of the largest circle that fits inside the hexagon. Find $d^2.$

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Solution:

All interior angles are 120°, so opposite sides are parallel. Attaching equilateral triangles to two opposite sides produces a parallelogram, which forces $AB + BC = DE + EF$ and $FA + AB = CD + DE.$ Hence $EF = 2$ and $FA = 16.$

Walking from one side to the opposite side along the two connecting sides shows that the distance between a pair of opposite sides is $\frac{\sqrt{3}}{2}$ times the sum of those two connecting sides: the strips have widths $\frac{\sqrt{3}}{2}(BC + CD) = 9\sqrt{3}$ between AB and DE, $\frac{\sqrt{3}}{2}(CD + DE) = 11\sqrt{3}$ between BC and EF, and $\frac{\sqrt{3}}{2}(DE + EF) = 7\sqrt{3}$ between CD and $FA.$ Any circle inside the hexagon fits in the narrowest strip, so $d \leq 7\sqrt{3}.$

A circle of diameter $7\sqrt{3}$ tangent to lines CD and FA can be centered so that it also touches DE exactly and has distances $6\sqrt{3}, 5\sqrt{3},$ and $\frac{11\sqrt{3}}{2}$ from lines EF, BC, and AB, all more than its radius $\frac{7\sqrt{3}}{2}$, so it fits inside the hexagon. Therefore $d = 7\sqrt{3}$ and $d^2 = 147.$

9. Find the number of four-element subsets of $\{1, 2, 3, 4, \ldots, 20\}$ with the property that two distinct elements of the subset have a sum of 16, and two distinct elements of the subset have a sum of 24. For example, $\{3, 5, 13, 19\}$ and $\{6, 10, 20, 18\}$ are two such subsets.

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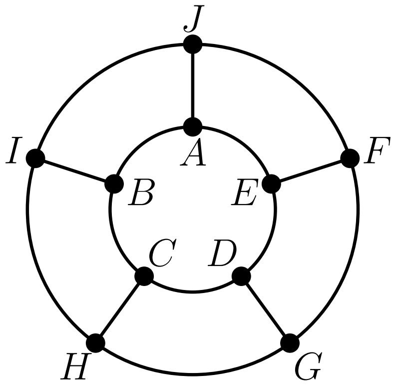
Solution:

The pairs of distinct elements summing to 16 are $\{1, 15\}, \{2, 14\}, \ldots, \{7, 9\}$ (seven pairs), and those summing to 24 are $\{4, 20\}, \{5, 19\}, \ldots, \{11, 13\}$ (eight pairs). First count subsets containing a 16-pair and a 24-pair that are disjoint. Of the $7 \cdot 8 = 56$ combinations, the ones sharing an element x require $16 - x, x$, and $24 - x$ all to be valid, which happens for the 10 values $x \in \{4, \ldots, 15\}$ other than 8 and 12. No four-element set arises from two different disjoint combinations (a second decomposition would force a 16-pair to coincide with a 24-pair), so this case gives $56 - 10 = 46$ subsets.

In the remaining subsets every 16-pair meets every 24-pair, so some center a has both $b = 16 - a$ and $c = 24 - a$ in the subset. There are 10 possible centers ($a \in \{4, \ldots, 15\}$ with $a \neq 8, 12$), and the fourth element can be any of the 17 remaining numbers, giving 170 center-subset counts. Exactly 6 subsets admit two centers and are counted twice: $\{1, 7, 9, 15\}, \{2, 6, 10, 14\}, \{3, 5, 11, 13\}, \{5, 11, 13, 19\}, \{6, 10, 14, 18\}$, and $\{7, 9, 15, 17\}$. This case gives $170 - 6 = 164$ subsets, none of which contain disjoint pairs.

The total is $46 + 164 = 210$.

10. The wheel shown below consists of two circles and five spokes, with a label at each point where a spoke meets a circle. A bug walks along the wheel, starting at point A. At every step of the process, the bug walks from one labeled point to an adjacent labeled point. Along the inner circle the bug only walks in a counterclockwise direction, and along the outer circle the bug only walks in a clockwise direction. For example, the bug could travel along the path $AJABCHCHIJA$, which has 10 steps. Let n be the number of paths with 15 steps that begin and end at point A. Find the remainder when n is divided by 1000.



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Solution:

From any inner point the bug has exactly two moves, counterclockwise along the inner circle or outward along a spoke; from any outer point it has exactly two, clockwise along the outer circle or inward along a spoke. Call a move X if it is counterclockwise or inward and Y if it is clockwise or outward. Then every string in $\{X, Y\}^{15}$ describes exactly one 15-step path from A.

A step arrives on the inner circle exactly when it is an X, so the path ends on the inner circle exactly when its last move is an X; in that case the numbers of inward and outward moves are equal. Measuring angular position in fifths of a turn (counterclockwise $+1$, clockwise -1, spokes 0), the path returns to A exactly when it ends on the inner circle and the net rotation is a multiple of 5, that is, when the last move is X and $\#X - \#Y \equiv 0 \pmod{5}$. With 15 moves this means the number of X s is $5, 10,$ or 15.

Fixing the last move as X, the first 14 moves contain $4, 9,$ or 14 Xs, so

$$n = \binom{14}{4} + \binom{14}{9} + \binom{14}{14} = 1001 + 2002 + 1 = 3004,$$

and the remainder is 4.

11. Find the least positive integer n such that when 3^n is written in base 143, its two rightmost digits in base 143 are 01.

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Solution:

The last two base-143 digits are 01 exactly when $3^n \equiv 1 \pmod{143^2}$, and since $143^2 = 11^2 \cdot 13^2$, this holds exactly when $3^n \equiv 1$ modulo both 11^2 and 13^2.

Modulo $121 : 3^5 = 243 = 2 \cdot 121 + 1 \equiv 1$, and since 5 is prime and $3 \not\equiv 1$, the order of 3 is exactly 5. Modulo 169 : the order of 3 modulo 13 is 3, so the order modulo 169 is a multiple of 3. Writing $3^3 = 27 = 1 + 26$ and noting $26^2 = 4 \cdot 169 \equiv 0 \pmod{169}$, the binomial theorem gives $3^{3k} = (1+26)^k \equiv 1 + 26k \pmod{169}$, which is 1 exactly when $13 \mid k$. So the order of 3 modulo 169 is 39.

Therefore n must be a common multiple of 5 and 39, and the least is $\operatorname{lcm}(5, 39) = 195$.

12. For each subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

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Solution:

The set U contains six elements in each residue class modulo 3. If T contains a elements $\equiv 1$ and b elements $\equiv 2 \pmod 3$, then $s(T) \equiv a + 2b \equiv a - b \pmod 3$, so $3 \mid s(T)$ exactly when $a \equiv b \pmod 3$; the six multiples of 3 may be included freely, contributing a factor 2^6 to both the favorable and total counts.

By Vandermonde's identity, the number of ways to choose the as and bs with $a - b = 0$ is $\sum_a \binom{6}{a}^2 = \binom{12}{6} = 924$; with $a - b = \pm 3$ it is $2 \sum_a \binom{6}{a}\binom{6}{a-3} = 2\binom{12}{9} = 440$; and with $a - b = \pm 6$ it is 2. The favorable choices number $924 + 440 + 2 = 1366$ out of 2^{12}.

The probability is $\frac{1366}{4096} = \frac{683}{2048}$, which is in lowest terms since 683 is odd. Thus $m = 683$.

13. Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

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Solution:

Since AI_1 and AI_2 bisect angles BAX and XAC, $\angle I_1AI_2 = \frac{1}{2}\angle BAX + \frac{1}{2}\angle XAC = \frac{A}{2}$, a constant. Let $\alpha = \angle AXB$. The incenter angle formula gives $\angle AI_1B = 90^\circ + \frac{\alpha}{2}$, so the law of sines in $\triangle ABI_1$ yields $AI_1 = \frac{AB \sin \frac{B}{2}}{\cos \frac{\alpha}{2}}$, and similarly, since $\angle AXC = 180^\circ - \alpha$, $AI_2 = \frac{AC \sin \frac{C}{2}}{\sin \frac{\alpha}{2}}$.

Therefore

$$[\triangle AI_1I_2] = \frac{1}{2} AI_1 \cdot AI_2 \sin \frac{A}{2} = \frac{AB \cdot AC \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \alpha},$$

which is minimized when $\alpha = 90^\circ$, that is, when X is the foot of the altitude from A.

With $a = 32, b = 34, c = 30$, and $s = 48$, the half-angle formulas give $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{(s-a)(s-b)(s-c)}{abc}$, so the minimum area is

$$bc \cdot \frac{(s-a)(s-b)(s-c)}{abc} = \frac{16 \cdot 14 \cdot 18}{32} = 126.$$

14. Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E.

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Solution:

Group the vertices into classes $\mathcal{A} = \{S, P_1\}$, $\mathcal{B} = \{P_2, P_5\}$, and $\mathcal{C} = \{P_3, P_4\}$. Each vertex of $\mathcal{A}$ adjoins one vertex of $\mathcal{A}$ and one of $\mathcal{B}$; each vertex of $\mathcal{B}$ adjoins one of $\mathcal{A}$ and one of $\mathcal{C}$; and each vertex of $\mathcal{C}$ adjoins one of $\mathcal{B}$ and the absorbing vertex E. Hence if a_n, b_n, c_n count the n-jump paths from S that have not yet reached E and end in each class,

$$a_{n+1} = a_n + b_n, \qquad b_{n+1} = a_n + c_n, \qquad c_{n+1} = b_n,$$

and exactly c_n paths reach E for the first time on jump $n + 1$.

Starting from $(a_0, b_0, c_0) = (1, 0, 0)$, the values of c_n for $n = 0, 1, \ldots, 11$ are $0, 0, 1, 1, 3, 4, 9, 14, 28, 47, 89, 155.$

The number of sequences of at most 12 jumps ending at E is $c_0 + c_1 + \cdots + c_{11} = 351.$

15. David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin\varphi_A = \frac{2}{3}$, $\sin\varphi_B = \frac{3}{5}$, and $\sin\varphi_C = \frac{6}{7}$. All three quadrilaterals have the same area K, which can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

The four sticks are chords of the unit circle subtending fixed arcs $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + \delta = 360^\circ$. The three quadrilaterals are the three distinct cyclic orders of the sides: say A has arcs in order $\alpha, \beta, \gamma, \delta$; then B (order $\alpha, \gamma, \beta, \delta$) and C (order $\alpha, \beta, \delta, \gamma$) are the other two. The angle between the diagonals of a cyclic quadrilateral is half the sum of the arcs subtended by either pair of opposite sides, so $\sin\varphi_B = \sin\frac{\alpha+\beta}{2}$ and $\sin\varphi_C = \sin\frac{\alpha+\delta}{2} = \sin\frac{\beta+\gamma}{2}$.

In a circle of radius 1, a chord spanning an arc θ has length $2\sin\frac{\theta}{2}$. The diagonals of A span the arcs $\alpha+\beta$ and $\beta+\gamma$, so their lengths are $2\sin\varphi_B$ and $2\sin\varphi_C$. Hence

$$K = \frac{1}{2} d_1 d_2 \sin\varphi_A = 2\sin\varphi_A \sin\varphi_B \sin\varphi_C,$$

a formula symmetric in the three quadrilaterals, which is why all three areas are equal.

Therefore $K = 2 \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{6}{7} = \frac{24}{35}$, and $m + n = 24 + 35 = 59$.

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