2017 AIME II Solutions

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1. Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are subsets of neither $\{1, 2, 3, 4, 5\}$ nor $\{4, 5, 6, 7, 8\}$.

Solution:

There are $2^8 = 256$ subsets in all. The ones to exclude are those contained in $\{1, 2, 3, 4, 5\}$ (there are $2^5 = 32$) or contained in $\{4, 5, 6, 7, 8\}$ (another 32). A subset of both is exactly a subset of the intersection $\{4, 5\}$, and there are $2^2 = 4$ of those.

By inclusion-exclusion, $32 + 32 - 4 = 60$ subsets fail, so $256 - 60 = 196$ subsets have the required property.

2. Teams T_1, T_2, T_3, and T_4 are in the playoffs. In the semifinal matches, T_1 plays T_4, and T_2 plays T_3. The winners of those two matches will play each other in the final match to determine the champion. When T_i plays T_j, the probability that T_i wins is $\frac{i}{i+j}$, and the outcomes of all the matches are independent. The probability that T_4 will be the champion is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

To be champion, T_4 must first beat T_1, which happens with probability $\frac{4}{4+1} = \frac{4}{5}$. The other semifinal sends T_2 to the final with probability $\frac{2}{2+3} = \frac{2}{5}$ and T_3 with probability $\frac{3}{5}$; in the final, T_4 beats T_2 with probability $\frac{4}{4+2} = \frac{2}{3}$ and beats T_3 with probability $\frac{4}{4+3} = \frac{4}{7}$.

The probability that T_4 is champion is therefore

$$\frac{4}{5}\left(\frac{2}{5}\cdot\frac{2}{3}+\frac{3}{5}\cdot\frac{4}{7}\right) = \frac{4}{5}\cdot\frac{64}{105} = \frac{256}{525}.$$

Since $256 = 2^8$ and $525 = 3\cdot 5^2\cdot 7$, this is in lowest terms, and $p+q = 256+525 = 781$.

3. A triangle has vertices $A(0,0)$, $B(12,0)$, and $C(8,10)$. The probability that a randomly chosen point inside the triangle is closer to vertex B than to either vertex A or vertex C can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

The points closer to B than to A lie to the right of the perpendicular bisector of $\overline{AB}$, the line $x=6$. The points closer to B than to C lie below the perpendicular bisector of $\overline{BC}$, which passes through the midpoint $(10,5)$ with slope $\frac{2}{5}$ (the negative reciprocal of the slope $-\frac{5}{2}$ of BC): the line $y=\frac{2}{5}x+1$.

Inside the triangle, the favorable region is the quadrilateral with vertices $(6,0)$, $B(12,0)$, the midpoint $(10,5)$ of $\overline{BC}$, and $\left(6,\frac{17}{5}\right)$, where the two bisectors meet. Splitting it along the segment from $(6,0)$ to $(10,5)$, its area is

$$\frac{1}{2}\cdot\frac{17}{5}\cdot 4+\frac{1}{2}\cdot 6\cdot 5=\frac{34}{5}+15=\frac{109}{5}.$$

The triangle has area $\frac{1}{2}\cdot 12\cdot 10=60$, so the probability is $\frac{109/5}{60}=\frac{109}{300}$, and $p+q=109+300=409$.

4. Find the number of positive integers less than or equal to 2017 whose base-three representation contains no digit equal to 0.

Solution:

A positive integer has no 0 in base three exactly when every digit is 1 or 2. For $k = 1, 2, \ldots, 6$ there are 2^k such k-digit numbers, and all of them are at most $222222_3 = 728 < 2017$.

Since $2017 = 2202201_3$, a seven-digit string of 1s and 2s is at most 2017 exactly when it begins with $11, 12,$ or 21 : any string beginning 22 already beats 2202201_3 at the third digit, since its digits are nonzero. That gives $3 \cdot 2^5 = 96$ seven-digit numbers.

The total is $2 + 4 + 8 + 16 + 32 + 64 + 96 = 222.$

5. A set contains four numbers. The six pairwise sums of distinct elements of the set, in no particular order, are $189, 320, 287, 234, x,$ and y. Find the greatest possible value of $x + y$.

Solution:

Let the set be $\{a, b, c, d\}$ with total $s = a + b + c + d$. The six pairwise sums come in three complementary pairs:

$$(a+b)+(c+d)=(a+c)+(b+d)=(a+d)+(b+c)=s.$$

No two of the pairings of $189, 320, 287, 234$ into two pairs give equal totals ($509 \neq 521, 476 \neq 554, 423 \neq 607$), so x and y are not paired with each other; each is paired with a given sum, and the remaining two given sums are paired together. Adding all six values, $x + y = 3s - (189 + 320 + 287 + 234) = 3s - 1030,$ where s is the sum of two of the given numbers.

The largest choice is $s = 320 + 287 = 607,$ giving $x + y = 3 \cdot 607 - 1030 = 791.$ This is attained by the set $\{51.5,\ 137.5,\ 182.5,\ 235.5\},$ whose pairwise sums are $189, 234, 287, 320, 373,$ and $418,$ with $373 + 418 = 791.$

6. Find the sum of all positive integers n such that $\sqrt{n^2 + 85n + 2017}$ is an integer.

Solution:

Suppose $n^2 + 85n + 2017 = m^2$ for a positive integer m. Multiplying by 4 and completing the square gives $(2n+85)^2 + 843 = 4m^2$, so

$$(2m - 2n - 85)(2m + 2n + 85) = 843 = 3 \cdot 281,$$

where 281 is prime. Both factors are positive with the second one larger, so either $2m - 2n - 85 = 1$ and $2m + 2n + 85 = 843$, or $2m - 2n - 85 = 3$ and $2m + 2n + 85 = 281$.

The first system gives $m = 211$ and $n = 168$, and indeed $168^2 + 85 \cdot 168 + 2017 = 44521 = 211^2$. The second gives $m = 71$ and $n = 27$, with $27^2 + 85 \cdot 27 + 2017 = 5041 = 71^2$.

The requested sum is $168 + 27 = 195$.

7. Find the number of integer values of k in the closed interval $[-500, 500]$ for which the equation $\log(kx) = 2\log(x+2)$ has exactly one real solution.

Solution:

The equation requires $x + 2 > 0$ and $kx > 0$, and under those restrictions it is equivalent to $kx = (x+2)^2$, that is, $x^2 + (4-k)x + 4 = 0$.

For $k < 0$ the restrictions force $-2 < x < 0$. On this interval kx decreases from $-2k > 0$ to 0 while $(x+2)^2$ increases from 0 to 4, so the graphs cross exactly once. Hence every one of the 500 negative values of k works, while $k = 0$ makes $\log(kx)$ undefined.

For $k > 0$ the restrictions force $x > 0$. The quadratic has root product 4, so any real roots have the same sign, and the discriminant $(4-k)^2 - 16 = k(k-8)$ is negative for $0 < k < 8$. When $k > 8$ there are two distinct positive roots (root sum $k - 4 > 0$), giving two solutions; only $k = 8$ gives exactly one solution, the double root $x = 2$. In total $500 + 1 = 501$ values of k work.

8. Find the number of positive integers n less than 2017 such that

$$1+n+\frac{n^2}{2!}+\frac{n^3}{3!}+\frac{n^4}{4!}+\frac{n^5}{5!}+\frac{n^6}{6!}$$

is an integer.

Solution:

Multiplying by $6! = 720$, the sum is an integer exactly when

$$720 \mid n^6 + 6n^5 + 30n^4 + 120n^3 + 360n^2.$$

If n were odd, every term except n^6 would be even, making the total odd. If $3 \nmid n$, then modulo 3 every term except n^6 vanishes while $n^6 \equiv 1 \pmod 3$. So n must be a multiple of 6.

Write $n = 6k$. Then $30n^4 = 720 \cdot 54k^4$, $120n^3 = 720 \cdot 36k^3$, and $360n^2 = 720 \cdot 18k^2$ are all divisible by 720, while $n^6 + 6n^5 = 6^6k^5(k+1)$. Since $6^6 = 2^63^6$ supplies the factors 2^4 and 3^2 of $720 = 2^4 \cdot 3^2 \cdot 5$, the condition reduces to $5 \mid k(k+1)$, that is, $k \equiv 0$ or $4 \pmod 5$.

For $n = 6k < 2017$ we need $1 \le k \le 336$. That range contains 67 multiples of 5 and 67 values with $k \equiv 4 \pmod 5$, so there are $67 + 67 = 134$ such n.

9. A special deck of cards contains 49 cards, each labeled with a number from 1 to 7 and colored with one of seven colors. Each number-color combination appears on exactly one card. Sharon will select a set of eight cards from the deck at random. Given that she gets at least one card of each color and at least one card with each number, the probability that Sharon can discard one of her cards and *still* have at least one card of each color and at least one card with each number is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

Since the eight cards cover all seven numbers and all seven colors, exactly one number and exactly one color appear twice. Sharon can discard a card exactly when a single card carries both the repeated number and the repeated color: that card is then the unique discardable one, while if no card carries both, removing any card loses a number or a color.

Hands of the first type consist of a rainbow set of seven cards — one of each number and each color, which is one of $7!$ permutation patterns — plus any of the remaining 42 cards, and every such hand arises exactly once this way: $7! \cdot 42 = 211680$ hands. For the second type, choose the repeated number (7 ways) and the two colors of its cards ($\binom{7}{2} = 21$ ways); the repeated color must be one of the other 5 colors, and the numbers of its two cards come from the remaining 6 numbers ($\binom{6}{2} = 15$ ways); finally match the last four numbers to the last four colors ($4! = 24$ ways). That is $7 \cdot 21 \cdot 5 \cdot 15 \cdot 24 = 264600$ hands.

The probability is $\frac{211680}{211680+264600} = \frac{4}{9}$, so $p+q = 4+9 = 13.$

10. Rectangle $ABCD$ has side lengths $AB = 84$ and $AD = 42$. Point M is the midpoint of $\overline{AD}$, point N is the trisection point of $\overline{AB}$ closer to A, and point O is the intersection of $\overline{CM}$ and $\overline{DN}$. Point P lies on the quadrilateral $BCON$, and $\overline{BP}$ bisects the area of $BCON$. Find the area of $\triangle CDP$.

Solution:

Place $A = (0,0), B = (84,0), C = (84,42), D = (0,42)$, so $M = (0,21)$ and $N = (28,0)$. Line CM is $y = \frac{x}{4} + 21$ and line DN is $y = 42 - \frac{3x}{2}$, which meet at $O = (12,24)$.

By the Shoelace Formula, quadrilateral $BCON$ has area 2184, so each half must have area 1092. Triangle BCO alone has area $\frac{1}{2} \cdot 42 \cdot 72 = 1512 > 1092$ (base $\overline{BC}$, and O is at horizontal distance 72 from it), so the bisecting segment ends at a point P on $\overline{CO}$. For $[BPC] = 1092$, the distance d from P to line BC must satisfy $\frac{1}{2} \cdot 42 \cdot d = 1092$, so $d = 52$, giving P the x-coordinate $84 - 52 = 32$. Since P lies on line CO, which is $y = \frac{x}{4} + 21$, we get $P = (32, 29)$.

Triangle CDP has base $CD = 84$ on the line $y = 42$ and height $42 - 29 = 13$, so its area is $\frac{1}{2} \cdot 84 \cdot 13 = 546$.

11. Five towns are connected by a system of roads. There is exactly one road connecting each pair of towns. Find the number of ways there are to make all the roads one-way in such a way that it is still possible to get from any town to any other town using the roads (possibly passing through other towns on the way).

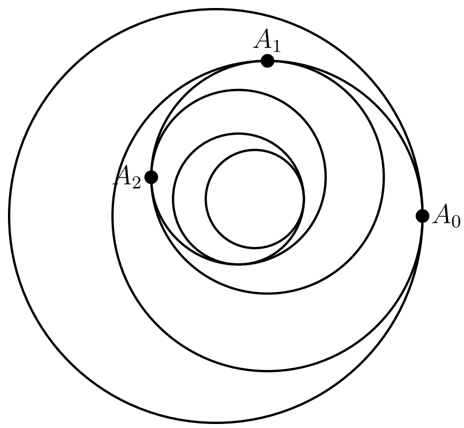
Solution:

The assignment works if and only if no town has all four roads inbound or all four outbound. One direction is clear: an all-inbound town cannot be left, and an all-outbound town cannot be reached. Conversely, suppose every town has an inbound and an outbound road, yet town B cannot be reached from town A. Let S be the set of towns reachable from A (including A) and T the set of towns from which B is reachable (including B). These sets are disjoint, every outbound road of a town in S stays inside S, and every inbound road of a town in T comes from inside T. Since A has an outbound road, $|S| \geq 2$, and similarly $|T| \geq 2$; as $|S| + |T| \leq 5$, one of the two sets has exactly two towns. If $S = \{A, X\}$, the outbound roads of A and of X must both stay inside S, forcing the single road between them to point both ways — a contradiction (and $|T| = 2$ is symmetric).

Now count assignments with a bad town among the $2^{10} = 1024$ total. Choosing a town to be all-outbound (5 ways) and orienting the remaining $\binom{4}{2} = 6$ roads freely gives $5 \cdot 2^6 = 320$ assignments, and there can be at most one all-outbound town. Similarly 320 assignments have an all-inbound town. Assignments with both are counted twice: choose the all-outbound town (5), the all-inbound town (4), and the other 3 roads freely, $5 \cdot 4 \cdot 2^3 = 160$. So $320 + 320 - 160 = 480$ assignments fail.

The number that work is $1024 - 480 = 544$.

12. Circle C_0 has radius 1, and the point A_0 is a point on the circle. Circle C_1 has radius $r < 1$ and is internally tangent to C_0 at point A_0. Point A_1 lies on circle C_1 so that A_1 is located 90° counterclockwise from A_0 on C_1. Circle C_2 has radius r^2 and is internally tangent to C_1 at point A_1. In this way a sequence of circles $C_1, C_2, C_3, \ldots$ and a sequence of points on the circles $A_1, A_2, A_3, \ldots$ are constructed, where circle C_n has radius r^n and is internally tangent to circle C_{n-1} at point A_{n-1}, and point A_n lies on C_n 90° counterclockwise from point A_{n-1}, as shown in the figure below. There is one point B inside all of these circles. When $r = \frac{11}{60}$, the distance from the center of C_0 to B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution:

Work in the complex plane with C_0 centered at $O_0 = 0$ and $A_0 = 1$, and let O_n be the center of C_n. Inductively, $A_n = O_n + r^n i^n$: this holds for $n = 0$, and since C_{n+1} is internally tangent to C_n at A_n, its center is $O_{n+1} = A_n - r^{n+1} i^n$; then A_n sits in direction i^n from O_{n+1}, so rotating 90° counterclockwise gives $A_{n+1} = O_{n+1} + r^{n+1} i^{n+1}$.

Therefore $O_{n+1} - O_n = \left(r^n - r^{n+1}\right) i^n = (1 - r)(ir)^n$. The circles are nested, and their radii shrink to 0, so the common point B is the limit of the centers:

$$B = (1 - r) \sum_{n=0}^{\infty} (ir)^n = \frac{1 - r}{1 - ir},$$

at distance $\frac{1-r}{|1-ir|} = \frac{1-r}{\sqrt{1+r^2}}$ from the origin.

For $r = \frac{11}{60}$ this equals $\frac{49/60}{\sqrt{3721}/60} = \frac{49}{61}$, so $m + n = 49 + 61 = 110.$

13. For each integer $n \geq 3$, let $f(n)$ be the number of 3-element subsets of the vertices of a regular n-gon that are the vertices of an isosceles triangle (including equilateral triangles). Find the sum of all values of n such that $f(n+1) = f(n) + 78.$

Solution:

Count isosceles triangles by apex. For a vertex P of a regular n-gon, the isosceles triangles whose two equal sides meet at P have their other two vertices symmetric about the diameter through P, giving $\lfloor (n-1)/2 \rfloor$ such pairs. Summing over all n vertices counts each non-equilateral isosceles triangle once (it has one apex) and each equilateral triangle three times; equilateral triangles exist exactly when $3 \mid n$, and then there are $n/3$ of them. Hence $f(n) = n\lfloor (n-1)/2 \rfloor$, minus $2n/3$ when $3 \mid n$.

Writing $n = 6k + j$ and computing $f(n+1) - f(n)$ in each residue class gives $13k$ for $j = 0, 3k$ for $j = 1, 5k + 1$ for $j = 2, 7k + 3$ for $j = 3, 9k + 6$ for $j = 4,$ and $-(k+2)$ for $j = 5$. Setting each equal to $78 : 13k = 78$ gives $k = 6, n = 36; 3k = 78$ gives $k = 26, n = 157; 9k + 6 = 78$ gives $k = 8, n = 52;$ and the other three cases have no positive integer solutions.

The sum of all such n is $36 + 157 + 52 = 245.$

14. A $10 \times 10 \times 10$ grid of points consists of all points in space of the form (i, j, k), where $i, j,$ and k are integers between 1 and $10,$ inclusive. Find the number of different lines that contain exactly 8 of these points.

Solution:

Take a primitive direction vector (a, b, c) for the line. Any nonzero component of absolute value 2 or more limits the line to at most 5 grid points, so every component is 0 or $\pm 1.$ Lines parallel to a coordinate axis contain 10 points, never $8.$ If exactly one component is $0,$ the line lies in one of the 30 planes parallel to a face of the cube (3 orientations, 10 positions), and within that 10×10 grid it is a diagonal of slope ± 1 shifted off center; the shift by 2 in either direction from each of the two main diagonals gives exactly 8 points. That is 4 lines per plane, and each lies in only one of the 30 planes: $4 \cdot 30 = 120$ lines.

Otherwise the direction is one of the four space-diagonal directions $(1, \pm 1, \pm 1)$ up to sign; by symmetry, count lines parallel to $(1, 1, 1)$ and multiply by $4.$ Such a line $(d + t,\ e + t,\ f + t)$ meets the grid in $10 - (\max(d, e, f) - \min(d, e, f))$ points, so exactly 8 points means $\max - \min = 2.$ Normalizing the base point so that $\min(d, e, f) = 1,$ we need (d, e, f) with entries in $\{1, 2, 3\}$ using both 1 and $3:$ there are $27 - 8 - 8 + 1 = 12$ of them, hence 12 lines per direction and $4 \cdot 12 = 48$ in all.

The total is $120 + 48 = 168.$

15. Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point X in space, define $f(X) = AX + BX + CX + DX$. The least possible value of $f(X)$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

Solution:

Let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$. The medians from C and from D to $\overline{AB}$ are equal, since triangles ABC and BAD are congruent by SSS; by the median length formula, $4MD^2 = 2 \cdot 28^2 + 2 \cdot 44^2 - 52^2 = 2736$, so $MC^2 = MD^2 = 684$. Likewise $NA = NB$. Then MN, as a median of the isosceles triangles MCD and NAB, is perpendicular to both $\overline{AB}$ and $\overline{CD}$, so the 180° rotation about line MN swaps $A \leftrightarrow B$ and $C \leftrightarrow D$. Also $MN^2 = MD^2 - ND^2 = 684 - 26^2 = 8$.

For any point X, let X' be its image under this rotation, and let Q be the midpoint of $\overline{XX'}$, which lies on line MN. Then $BX = AX'$ and $CX = DX'$, so

$$f(X) = (AX + AX') + (DX + DX') \geq 2AQ + 2DQ = f(Q),$$

because a median of a triangle is at most half the sum of the two adjacent sides. So it suffices to minimize $f(Q) = 2(AQ + DQ)$ over points Q on line MN.

Rotate D about line MN into the plane of A and line MN, on the opposite side of MN from A, landing at D' with $ND' = ND = 26$. For Q on line MN, $AQ + DQ = AQ + D'Q \geq AD'$, with equality where segment $\overline{AD'}$ crosses MN. Since $AM \perp MN$ and $D'N \perp MN$ with $AM = 26$ and $ND' = 26$,

$$AD'^2 = (AM + ND')^2 + MN^2 = 52^2 + 8 = 2712 = 4 \cdot 678.$$

Hence the minimum of f is $2AD' = 4\sqrt{678}$, and since $678 = 2 \cdot 3 \cdot 113$ is squarefree, $m + n = 4 + 678 = 682$.

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