# 2016 AIME II Solutions

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**1.** Initially Alex, Betty, and Charlie had a total of $444$ peanuts. Charlie had the most peanuts, and Alex had the least. The three numbers of peanuts that each person had form a geometric progression. Alex eats $5$ of his peanuts, Betty eats $9$ of her peanuts, and Charlie eats $25$ of his peanuts. Now the three numbers of peanuts that each person has form an arithmetic progression. Find the number of peanuts Alex had initially.

## Solution:

After the eating, $444 - 5 - 9 - 25 = 405$ peanuts remain, and the three amounts form an arithmetic progression, so the middle amount, Betty's, is $\frac{405}{3} = 135.$ Hence Betty started with $135 + 9 = 144$ peanuts.

The starting amounts form a geometric progression, so they are $\frac{144}{r}$, $144,$ and $144r$ with $r > 1$ (Charlie had the most and Alex the least). Then

$$\frac{144}{r} + 144 + 144r = 444,$$

which simplifies to $12r^2 - 25r + 12 = 0,$ with roots $r = \frac{4}{3}$ and $r = \frac{3}{4}$; since $r > 1,$ we take $r = \frac{4}{3}.$

So Alex initially had $144 \cdot \frac{3}{4} = 108$ peanuts. (Check: after eating, the amounts $103, 135, 167$ increase by $32$ each.)

**2.** There is a $40\%$ chance of rain on Saturday and a $30\%$ chance of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. The probability that it rains at least one day this weekend is $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a+b$.

## Solution:

Let $p$ be the probability that it rains on Sunday given a dry Saturday; given a rainy Saturday it is $2p$. The overall Sunday chance gives

$$0.4(2p)+0.6p=0.3,$$

so $1.4p=0.3$ and $p=\frac{3}{14}$.

The weekend is completely dry exactly when Saturday is dry and then Sunday is dry: $\frac{3}{5}\left(1-\frac{3}{14}\right)=\frac{3}{5}\cdot\frac{11}{14}=\frac{33}{70}$. So the probability of rain on at least one day is $1-\frac{33}{70}=\frac{37}{70}$, which is in lowest terms, and $a+b=37+70=107.$

**3.** Let $x, y$, and $z$ be real numbers satisfying the system

$$\log_2(xyz - 3 + \log_5 x) = 5$$

$$\log_3(xyz - 3 + \log_5 y) = 4$$

$$\log_4(xyz - 3 + \log_5 z) = 4.$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

## Solution:

Exponentiating each equation gives $xyz - 3 + \log_5 x = 2^5$ and similarly for the others, so

$$xyz + \log_5 x = 35, \qquad xyz + \log_5 y = 84, \qquad xyz + \log_5 z = 259.$$

Write $x = 5^a, y = 5^b, z = 5^c$, so that $xyz = 5^{a+b+c}$.

Adding the three equations gives $3 \cdot 5^s + s = 378$, where $s = a + b + c$. The left side is strictly increasing in $s$, and $s = 3$ works since $375 + 3 = 378$, so $s = 3$ and $5^s = 125$.

Then $a = 35 - 125 = -90$, $b = 84 - 125 = -41$, and $c = 259 - 125 = 134$, so $|a| + |b| + |c| = 90 + 41 + 134 = 265$.

**4.** An $a \times b \times c$ rectangular box is built from $a \cdot b \cdot c$ unit cubes. Each unit cube is colored red, green, or yellow. Each of the $a$ layers of size $1 \times b \times c$ parallel to the $(b \times c)$-faces of the box contains exactly $9$ red cubes, exactly $12$ green cubes, and some yellow cubes. Each of the $b$ layers of size $a \times 1 \times c$ parallel to the $(a \times c)$-faces of the box contains exactly $20$ green cubes, exactly $25$ yellow cubes, and some red cubes. Find the smallest possible volume of the box.

## Solution:

Each $1 \times b \times c$ layer has exactly $9$ red and $12$ green cubes, hence exactly $bc - 21$ yellow; each $a \times 1 \times c$ layer has exactly $20$ green and $25$ yellow, hence exactly $ac - 45$ red. Counting green cubes in the whole box both ways gives $12a = 20b$, so $3a = 5b$. Counting yellow both ways gives $a(bc - 21) = 25b = 15a$, so $bc = 36$. Counting red both ways gives $b(ac - 45) = 9a$, and $\frac{9a}{b} = 15$, so $ac = 60$.

Thus $a = \frac{60}{c}$ and $b = \frac{36}{c}$ are positive integers, so $c$ divides $\gcd(60, 36) = 12$, and the volume is $abc = \frac{60 \cdot 36}{c} = \frac{2160}{c}$, smallest when $c = 12$ : volume $180$ with $(a, b, c) = (5, 3, 12)$.

This is achievable: color every $1 \times 3 \times 12$ layer with three identical rows RRRGGGGYYYYY. Then each $1 \times 3 \times 12$ layer has $9$ red, $12$ green, and $15$ yellow cubes, and each $5 \times 1 \times 12$ layer has $15$ red, $20$ green, and $25$ yellow cubes. So the smallest possible volume is $180.$

**5.** Triangle $ABC_0$ has a right angle at $C_0$. Its side lengths are pairwise relatively prime positive integers, and its perimeter is $p$. Let $C_1$ be the foot of the altitude to $\overline{AB}$, and for $n \geq 2$, let $C_n$ be the foot of the altitude to $\overline{C_{n-2}B}$ in $\triangle C_{n-2}C_{n-1}B$. The sum $\sum_{n=1}^{\infty} C_{n-1}C_n = 6p$. Find $p$.

## Solution:

Let $a = BC_0, b = AC_0$, and $c = AB$. The altitude gives $C_0C_1 = \frac{ab}{c}$, and $\triangle C_0C_1B \sim \triangle AC_0B$ with ratio $\frac{a}{c}$. Each later altitude repeats this construction in a triangle scaled by $\frac{a}{c}$, so the segments $C_{n-1}C_n$ form a geometric series and

$$\sum_{n=1}^{\infty} C_{n-1}C_n = \frac{ab/c}{1 - a/c} = \frac{ab}{c-a} = 6p = 6(a+b+c).$$

Since $b^2 = c^2 - a^2 = (c-a)(c+a)$, we get $(c-a)(a+b+c) = (c-a)(c+a) + (c-a)b = b(b+c-a)$, so $ab = 6b(b+c-a)$, that is $7a = 6b + 6c$. Squaring $6c = 7a - 6b$ and using $36c^2 = 36a^2 + 36b^2$ gives $36a^2 = 49a^2 - 84ab$, hence $13a = 84b$.

Because the side lengths are pairwise relatively prime, $a = 84$ and $b = 13$, so $c = \frac{7 \cdot 84 - 6 \cdot 13}{6} = 85$, and indeed $13^2 + 84^2 = 85^2$. Then $p = 84 + 13 + 85 = 182$ (and $\frac{ab}{c-a} = \frac{84 \cdot 13}{1} = 1092 = 6p$ checks).

**6.** For polynomial $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$, define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then $\sum_{i=0}^{50} |a_i| = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

## Solution:

Every substituted power $x, x^3, x^5, x^7, x^9$ is odd, so $Q(-x) = P(-x)P(-x^3)P(-x^5)P(-x^7)P(-x^9)$. Since $P(-x) = 1 + \frac{1}{3}x + \frac{1}{6}x^2$ has only nonnegative coefficients, so does each factor $P(-x^k)$, and hence so does the product $Q(-x)$. The coefficient of $x^i$ in $Q(-x)$ is $(-1)^i a_i$, so $|a_i| = (-1)^i a_i$.

Therefore

$$\sum_{i=0}^{50} |a_i| = Q(-1) = P(-1)^5 = \left(1 + \frac{1}{3} + \frac{1}{6}\right)^5 = \left(\frac{3}{2}\right)^5 = \frac{243}{32},$$

and $m + n = 243 + 32 = 275.$

**7.** Squares $ABCD$ and $EFGH$ have a common center and $\overline{AB} \parallel \overline{EF}$. The area of $ABCD$ is $2016$, and the area of $EFGH$ is a smaller positive integer. Square $IJKL$ is constructed so that each of its vertices lies on a side of $ABCD$ and each vertex of $EFGH$ lies on a side of $IJKL$. Find the difference between the largest and smallest possible integer values for the area of $IJKL$.

## Solution:

If a square of side $t$ has its vertices on the sides of a concentric square of side $s$ and is tilted by angle $\theta$, each side of the outer square is split into pieces $t\cos\theta$ and $t\sin\theta$, so $s = t(\cos\theta + \sin\theta)$. This applies to $IJKL$ (side $t$) in $ABCD$ (side $s$) with some angle $\theta$. Since $\overline{EF} \parallel \overline{AB}$, square $EFGH$ (side $u$) makes the same angle $\theta$ with $IJKL$, so also $t = u(\cos\theta + \sin\theta)$.

Hence $\frac{s}{t} = \frac{t}{u}$, so the three areas form a geometric progression: the area of $EFGH$ equals $\frac{T^2}{2016}$, where $T$ is the area of $IJKL$. As $\theta$ ranges over $(0^\circ, 90^\circ)$ , the factor $(\cos\theta + \sin\theta)^2$ takes every value in $(1, 2]$, so $T = \frac{2016}{(\cos\theta+\sin\theta)^2}$ takes every value in $[1008, 2016)$ ($\theta = 0^\circ$ is excluded because $EFGH$ is smaller than $ABCD$). For $\frac{T^2}{2016}$ to be an integer, $2016 = 2^5 \cdot 3^2 \cdot 7$ must divide $T^2$, which forces $2^3 \cdot 3 \cdot 7 = 168$ to divide $T$.

The multiples of $168$ in $[1008, 2016)$ run from $1008$ to $1848$, and each is attained by an appropriate $\theta$, with the area of $EFGH$ then a positive integer less than $2016$. The difference is $1848 - 1008 = 840$.

**8.** Find the number of sets $\{a, b, c\}$ of three distinct positive integers with the property that the product of $a, b,$ and $c$ is equal to the product of $11, 21, 31, 41, 51,$ and $61.$

## Solution:

Count ordered triples $(a, b, c)$ with $abc = 11 \cdot 21 \cdot 31 \cdot 41 \cdot 51 \cdot 61 = 3^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot 61 = N.$ Each of the six primes $7, 11, 17, 31, 41, 61$ appears once and can go to any of the three values: $3^6$ ways. The two factors of $3$ can be split among the three values in $\binom{4}{2} = 6$ ways. That gives $6 \cdot 3^6 = 4374$ ordered triples.

If two of the three values were equal, their common value $v$ would satisfy $v^2 \mid N$, so $v = 1$ or $v = 3.$ This produces the triples with values $\{1, 1, N\}$ and $\{3, 3, \frac{N}{9}\}$, each in $3$ orders, for $6$ ordered triples in all (all three equal is impossible). The remaining $4374 - 6 = 4368$ ordered triples have distinct entries, and each set $\{a, b, c\}$ is counted $3! = 6$ times.

So the number of sets is $\frac{4368}{6} = 728.$

**9.** The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer $k$ such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find $c_k$.

## Solution:

Write $a_n = 1 + (n-1)d$ and $b_n = r^{n-1}$ with integers $d \geq 1$ and $r \geq 2$. Since $c_1 = 2 < 100$, we have $k \geq 3$, and the two conditions read

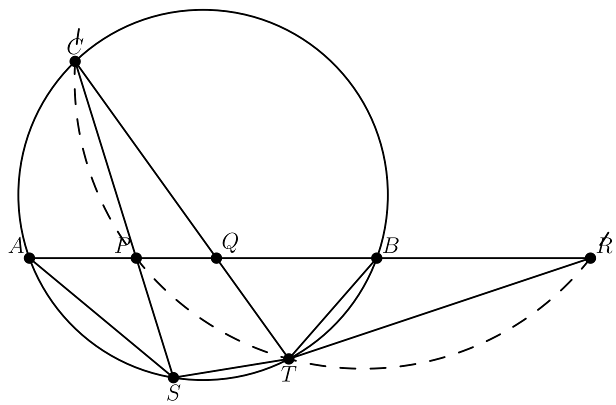
$$(k-2)d + r^{k-2} = 99, \qquad kd + r^k = 999.$$

Subtracting, $2d + r^{k-3}(r-1)r(r+1) = 900$. The product of three consecutive integers is divisible by $3$, so $3 \mid 2d$, hence $3 \mid d$. Then $(k-2)d + r^{k-2} = 99$ forces $3 \mid r^{k-2}$, so $3 \mid r$. The bounds $r^{k-2} \leq 98$ and $r^k \leq 998$ leave only $(r, k) = (3,3), (3,4), (3,5), (3,6), (6,3), (9,3)$.

Testing each against $(k-2)d = 99 - r^{k-2}$ and $kd = 999 - r^k$, only $(r,k) = (9,3)$ gives a consistent value, $d = 90$. Then $c_3 = 1 + 2 \cdot 90 + 9^2 = 262$.

**10.** Triangle $ABC$ is inscribed in circle $\omega$. Points $P$ and $Q$ are on side $\overline{AB}$ with $AP < AQ$. Rays $CP$ and $CQ$ meet $\omega$ again at $S$ and $T$ (other than $C$), respectively. If $AP = 4, PQ = 3, QB = 6, BT = 5$, and $AS = 7$, then $ST = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

## Solution:




By Power of a Point at $Q$ in $\omega$, $QC \cdot QT = QA \cdot QB = 7 \cdot 6 = 42$. Extend $\overline{AB}$ beyond $B$ to the point $R$ with $BR = 8$, so that $QR = QB + BR = 14$ and $QP \cdot QR = 3 \cdot 14 = 42 = QC \cdot QT$. By the converse of Power of a Point, $C, P, T$, and $R$ are concyclic.

In circle $CPTR$, $\angle BRT = \angle PRT = \angle PCT$, and in $\omega$, $\angle PCT = \angle SCT = \angle SAT$ (both subtend arc $ST$). Also $ASTB$ is cyclic, so the exterior angle of the quadrilateral at $B$ equals the opposite interior angle: $\angle RBT = \angle AST$. Hence $\triangle AST \sim \triangle RBT$.

Therefore $\frac{ST}{BT} = \frac{AS}{RB}$, so $ST = 5 \cdot \frac{7}{8} = \frac{35}{8}$, and $m + n = 35 + 8 = 43$.

**11.** For positive integers $N$ and $k$, define $N$ to be $k$-*nice* if there exists a positive integer $a$ such that $a^k$ has exactly $N$ positive divisors. Find the number of positive integers less than $1000$ that are neither $7$-nice nor $8$-nice.

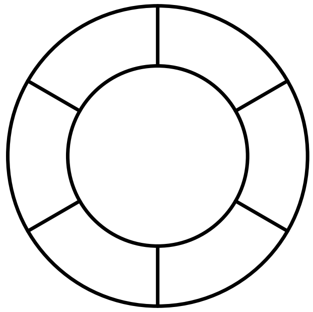
## Solution:

If $a = p_1^{m_1} \cdots p_t^{m_t}$, then $a^k$ has $(km_1 + 1)(km_2 + 1) \cdots (km_t + 1)$ positive divisors, and each factor is $\equiv 1 \pmod{k}$, so the product is too. Conversely, if $N = km + 1$, then $a = p^m$ gives $a^k = p^{km}$ with exactly $N$ divisors. So $N$ is $k$-nice exactly when $N \equiv 1 \pmod{k}$.

Among $1, 2, \ldots, 999$ there are $143$ integers $\equiv 1 \pmod{7}$ (namely $1, 8, \ldots, 995$) and $125$ integers $\equiv 1 \pmod{8}$ (namely $1, 9, \ldots, 993$). Since $\operatorname{lcm}(7, 8) = 56$, there are $18$ integers $\equiv 1 \pmod{56}$ (namely $1, 57, \ldots, 953$). By inclusion-exclusion, $143 + 125 - 18 = 250$ of them are $7$-nice or $8$-nice.

Hence $999 - 250 = 749$ positive integers less than $1000$ are neither.

**12.** The figure below shows a ring made of six small sections which you are to paint on a wall. You have four paint colors available and will paint each of the six sections a solid color. Find the number of ways you can choose to paint the sections if no two adjacent sections can be painted with the same color.



## Solution:

Let $P_n$ be the number of valid paintings of a ring of $n$ sections. Cutting a ring open between two adjacent sections shows that ring paintings correspond exactly to rows of $n$ sections with adjacent colors different *and* the two end colors different. A row of $n$ sections with adjacent colors different can be painted in $4 \cdot 3^{n-1}$ ways, and the rows whose end colors match correspond, by merging the two end sections into one, to ring paintings of $n-1$ sections. Hence

$$P_n + P_{n-1} = 4 \cdot 3^{n-1}.$$

Three mutually adjacent sections give $P_3 = 4 \cdot 3 \cdot 2 = 24$, so $P_4 = 108 - 24 = 84$, then $P_5 = 324 - 84 = 240$, and finally $P_6 = 972 - 240 = 732$.

**13.** Beatrix is going to place six rooks on a $6 \times 6$ chessboard where both the rows and columns are labeled $1$ to $6$; the rooks are placed so that no two rooks are in the same row or the same column. The *value* of a square is the sum of its row number and column number. The *score* of an arrangement of rooks is the least value of any occupied square. The average score over all valid configurations is $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

## Solution:

There are $6! = 720$ arrangements, and every score lies between $2$ and $7$. Let $b_n$ be the number of arrangements with score at least $n$. Since each score $s$ satisfies $s = 2 + \#\{n \geq 3 : s \geq n\}$, the total of all $720$ scores is

$$2 \cdot 720 + b_3 + b_4 + b_5 + b_6 + b_7.$$

Score $\geq n$ means no rook occupies a square with row + column $< n$. Place the rooks row by row. For $b_3$, only $(1, 1)$ is banned: $5 \cdot 5! = 600$. For $b_4$, row $1$ has $4$ allowed columns, then row $2$ has $4$ (column $1$ and the used column are excluded): $4 \cdot 4 \cdot 4! = 384$. Similarly $b_5 = 3 \cdot 3 \cdot 3 \cdot 3! = 162$, $b_6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2! = 32$, and $b_7 = 1$ (all rooks on the anti-diagonal).

The total is $1440 + 600 + 384 + 162 + 32 + 1 = 2619$, so the average is $\frac{2619}{720} = \frac{291}{80}$, and $p + q = 291 + 80 = 371$.

**14.** Equilateral $\triangle ABC$ has side length $600$. Points $P$ and $Q$ lie outside the plane of $\triangle ABC$ and are on opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a $120^\circ$ dihedral angle (the angle between the two planes). There is a point $O$ whose distance from each of $A, B, C, P$, and $Q$ is $d$. Find $d$.

## Solution:

Since $PA = PB = PC$ and $QA = QB = QC$, both $P$ and $Q$ lie on the line through the center $H$ of $\triangle ABC$ perpendicular to its plane, on opposite sides. Any point equidistant from $A, B, C$ also lies on that line, so $O$ is on it, and $OP = OQ = d$ makes $O$ the midpoint of $\overline{PQ}$, with $PQ = 2d$. Let $D$ be the midpoint of $\overline{AB}$ and $a = 600$; then $DH = \frac{a\sqrt{3}}{6}$ and $CH = \frac{a\sqrt{3}}{3}$. Since $\overline{PD} \perp \overline{AB}$ and $\overline{QD} \perp \overline{AB}$, the dihedral angle is $\angle PDQ = 120^\circ$; write $x = \angle PDH$ and $y = \angle QDH$, so $x + y = 120^\circ$.

Right triangles $PDH$ and $QDH$ give $PH = DH \tan x$ and $QH = DH \tan y$, so

$$2d = PQ = PH + QH = \frac{a\sqrt{3}}{6}(\tan x + \tan y).$$

Since $OC = OP = OQ = d$, point $C$ lies on the circle with diameter $\overline{PQ}$, so $\angle PCQ = 90^\circ$, and $H$ is the foot of the altitude from $C$ to the hypotenuse $\overline{PQ}$. Thus $CH^2 = PH \cdot QH$, which gives $\tan x \tan y = \frac{CH^2}{DH^2} = 4$.

By the tangent addition formula, $-\sqrt{3} = \tan 120^\circ = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\tan x + \tan y}{-3}$, so $\tan x + \tan y = 3\sqrt{3}$. Then $2d = \frac{a\sqrt{3}}{6} \cdot 3\sqrt{3} = \frac{3a}{2}$, so $d = \frac{3a}{4} = 450$.

**15.** For $1 \le i \le 215$ let $a_i = \frac{1}{2^i}$ and $a_{216} = \frac{1}{2^{215}}$. Let $x_1, x_2, \ldots, x_{216}$ be positive real numbers such that

$$\sum_{i=1}^{216} x_i = 1 \quad \text{and} \quad \sum_{1 \le i < j \le 216} x_i x_j = \frac{107}{215} + \sum_{i=1}^{216} \frac{a_i x_i^2}{2(1-a_i)}.$$

The maximum possible value of $x_2 = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

## Solution:

Since $\sum x_i = 1$, we have $2\sum_{i<j} x_i x_j = 1 - \sum x_i^2$. Doubling the given equation and rearranging,

$$1 - \sum_{i=1}^{216} x_i^2 = \frac{214}{215} + \sum_{i=1}^{216} \frac{a_i x_i^2}{1-a_i}, \qquad \text{so} \qquad \frac{1}{215} = \sum_{i=1}^{216}\left(1 + \frac{a_i}{1-a_i}\right) x_i^2 = \sum_{i=1}^{216} \frac{x_i^2}{1-a_i}.$$

Now $\sum a_i = \left(\frac{1}{2} + \cdots + \frac{1}{2^{215}}\right) + \frac{1}{2^{215}} = 1$, so $\sum(1-a_i) = 216 - 1 = 215$. By the Cauchy-Schwarz inequality,

$$1 = \left(\sum_{i=1}^{216} x_i\right)^2 \le \left(\sum_{i=1}^{216} \frac{x_i^2}{1-a_i}\right)\left(\sum_{i=1}^{216}(1-a_i)\right) = \frac{1}{215} \cdot 215 = 1.$$

Equality holds, so $x_i$ is proportional to $1-a_i$, forcing $x_i = \frac{1-a_i}{215}$. The only, hence maximum, possible value of $x_2$ is $\frac{1-\frac{1}{4}}{215} = \frac{3}{860}$, and $m+n = 3+860 = 863$.

Problems: https://live.poshenloh.com/past-contests/aime/2016II