2016 AIME I Solutions

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1. For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots .$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Solution:

The geometric series sums to $S(r) = \frac{12}{1-r}$. Therefore

$$S(a)S(-a) = \frac{12}{1-a} \cdot \frac{12}{1+a} = \frac{144}{1-a^2} = 2016,$$

so $\frac{1}{1-a^2} = 14$.

Adding the two sums over a common denominator,

$$S(a) + S(-a) = \frac{12}{1-a} + \frac{12}{1+a} = \frac{24}{1-a^2} = 24 \cdot 14 = 336.$$

2. Two dice appear to be standard dice with their faces numbered from 1 to 6, but each die is weighted so that the probability of rolling the number k is directly proportional to k. The probability of rolling a 7 with this pair of dice is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

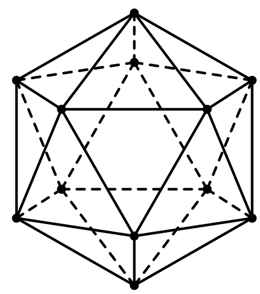
Solution:

Since $1 + 2 + \cdots + 6 = 21$, each die rolls k with probability $\frac{k}{21}$. A total of 7 arises from the pairs $(k, 7 - k)$ for $k = 1, \ldots, 6$, so its probability is

$$\frac{1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1}{21^2} = \frac{56}{441} = \frac{8}{63}.$$

Thus $m + n = 8 + 63 = 71$.

3. A *regular icosahedron* is a 20-faced solid where each face is an equilateral triangle and five triangles meet at every vertex. The regular icosahedron shown below has one vertex at the top, one vertex at the bottom, an upper pentagon of five vertices all adjacent to the top vertex and all in the same horizontal plane, and a lower pentagon of five vertices all adjacent to the bottom vertex and all in another horizontal plane. Find the number of paths from the top vertex to the bottom vertex such that each part of a path goes downward or horizontally along an edge of the icosahedron, and no vertex is repeated.



Solution:

Each vertex of the upper pentagon is adjacent to the top vertex, two upper-pentagon neighbors, and two vertices of the lower pentagon; each vertex of the lower pentagon is adjacent to two upper vertices, two lower-pentagon neighbors, and the bottom vertex. So a downward-or-horizontal path with no repeated vertex must descend to the upper pentagon, circle part of it in one direction, drop to the lower pentagon, circle part of it in one direction, and end at the bottom.

There are 5 choices for the first step down. On the upper pentagon the path can take $0, 1, 2, 3,$ or 4 horizontal steps, in either of two directions (a reversal would repeat a vertex), for $1 + 2 \cdot 4 = 9$ options. Then there are 2 edges down to the lower pentagon, again 9 horizontal options there, and 1 final step down.

The total is $5 \cdot 9 \cdot 2 \cdot 9 = 810.$

4. A right prism with height h has bases that are regular hexagons with sides of length $12.$ A vertex A of the prism and its three adjacent vertices are the vertices of a triangular pyramid. The dihedral angle (the angle between the two planes) formed by the face of the pyramid that lies in a base of the prism and the face of the pyramid that does not contain A measures 60°. Find h^2.

Solution:

The three vertices adjacent to A are its two neighbors B and C in the same hexagonal base and the vertex D directly above A, with $DA = h$ perpendicular to the base. The face in the base is ABC, and the face avoiding A is BCD; they meet along $\overline{BC}$.

Let E be the midpoint of $\overline{BC}$. Since $AB = AC = 12$ and $\angle BAC = 120^\circ$ (the interior angle of a regular hexagon), $\overline{AE} \perp \overline{BC}$ and $AE = 12\cos 60^\circ = 6$. Because $\overline{DA}$ is perpendicular to the base, $\overline{DE} \perp \overline{BC}$ as well, so the dihedral angle is $\angle DEA = 60^\circ$.

In right triangle DAE, $h = AE\tan 60^\circ = 6\sqrt{3}$, so $h^2 = 108.$

5. Anh read a book. On the first day she read n pages in t minutes, where n and t are positive integers. On the second day Anh read $n+1$ pages in $t+1$ minutes. Each day thereafter Anh read one more page than she read on the previous day, and it took her one more minute than on the previous day until she completely read the 374 page book. It took her a total of 319 minutes to read the book. Find $n+t$.

Solution:

Say Anh finished on day k. Summing the arithmetic progressions of pages and of minutes,

$$\frac{k(2n+k-1)}{2} = 374 \qquad \text{and} \qquad \frac{k(2t+k-1)}{2} = 319,$$

so $k(2n+k-1) = 748$ and $k(2t+k-1) = 638.$

Subtracting, $2k(n-t) = 110$, so $k(n-t) = 55.$ Thus k divides both 55 and $\gcd(748, 638) = 22$, so $k \mid 11$. Since the story spans more than one day, $k = 11.$

Then $2n + 10 = \frac{748}{11} = 68$ gives $n = 29$, and $2t + 10 = \frac{638}{11} = 58$ gives $t = 24.$ Hence $n + t = 29 + 24 = 53.$

6. In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect $\overline{AB}$ at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:

The incenter I lies on the bisector $\overline{CL}$, between C and L. In triangle ACI, the exterior angle at I gives $\angle DIA = \angle IAC + \angle ICA$. On the other hand, $\angle DAB = \angle DCB$ (both subtend arc DB) and $\angle DCB = \angle ICA$ (the bisector), so

$$\angle DAI = \angle DAB + \angle BAI = \angle ICA + \angle IAC = \angle DIA.$$

Hence triangle DAI is isosceles with $DA = DI = DL + LI = 5$.

Triangles DAL and DCA have a common angle at D, and $\angle DAL = \angle DAB = \angle DCB = \angle DCA$, so they are similar. Therefore $\frac{DA}{DC} = \frac{DL}{DA}$, giving $DC = \frac{DA^2}{DL} = \frac{25}{3}$.

Finally $IC = DC - DI = \frac{25}{3} - 5 = \frac{10}{3}$, so $p + q = 10 + 3 = 13$.

7. For integers a and b consider the complex number

$$\frac{\sqrt{ab + 2016}}{ab + 100} - \left(\frac{\sqrt{|a + b|}}{ab + 100}\right) i.$$

Find the number of ordered pairs of integers (a, b) such that this complex number is a real number.

Solution:

If $ab + 2016 \geq 0$, the first term is real, so the number is real exactly when $\sqrt{|a + b|} = 0$, that is $b = -a$. Then $ab + 2016 = 2016 - a^2 \geq 0$ forces $|a| \leq 44$, and the denominator $ab + 100 = 100 - a^2$ rules out $a = \pm 10$. That gives $89 - 2 = 87$ pairs.

If $ab + 2016 < 0$, then $\sqrt{ab + 2016} = i\sqrt{-ab - 2016}$, so the whole number is $\frac{\sqrt{-ab-2016}-\sqrt{|a+b|}}{ab+100}\, i$, which is real exactly when $-ab - 2016 = |a + b|$. Note $a + b = 0$ is impossible here since $a^2 = 2016$ has no integer solution. For $a + b > 0$ the equation becomes $ab + a + b + 2016 = 0$, that is $(a + 1)(b + 1) = -2015$, and for $a + b < 0$ it becomes $(a - 1)(b - 1) = -2015$.

Since $2015 = 5 \cdot 13 \cdot 31$ has 8 positive divisors, $(a + 1)(b + 1) = -2015$ has 16 ordered integer solutions, and $a + b > 0$ holds exactly when the positive factor is the larger in absolute value: 8 solutions. Symmetrically the other case gives 8 more. In all of these $ab + 100 = -1916 - |a + b| \neq 0$. The total is $87 + 8 + 8 = 103$.

8. For a permutation $p = (a_1, a_2, \ldots, a_9)$ of the digits $1, 2, \ldots, 9$, let $s(p)$ denote the sum of the three 3-digit numbers $a_1a_2a_3$, $a_4a_5a_6$, and $a_7a_8a_9$. Let m be the minimum value of $s(p)$ subject to the condition that the units digit of $s(p)$ is 0. Let n denote the number of permutations p with $s(p) = m$. Find $|m - n|$.

Solution:

By place value, $s(p) = 100(a_1 + a_4 + a_7) + 10(a_2 + a_5 + a_8) + (a_3 + a_6 + a_9)$, and all nine digits sum to 45. The units digit of $s(p)$ is 0 exactly when $a_3 + a_6 + a_9 = 10$ or 20. Writing $X = a_1 + a_4 + a_7$, if the units column sums to 10 then $s(p) = 100X + 10(35 - X) + 10 = 90X + 360 \geq 900$, while if it sums to 20 then $s(p) = 90X + 270 \geq 90 \cdot 6 + 270 = 810$. So $m = 810$, achieved exactly when $\{a_1, a_4, a_7\} = \{1, 2, 3\}$ and the units digits sum to 20.

The remaining digits $\{4, 5, 6, 7, 8, 9\}$ must split so the units triple sums to 20 : the possibilities are $\{4, 7, 9\}$, $\{5, 6, 9\}$, and $\{5, 7, 8\}$. Each of the 3 splits allows $3! \cdot 3! \cdot 3! = 216$ arrangements of the three columns, so $n = 3 \cdot 216 = 648$.

Therefore $|m - n| = |810 - 648| = 162$.

9. Triangle ABC has $AB = 40$, $AC = 31$, and $\sin A = \frac{1}{5}$. This triangle is inscribed in rectangle $AQRS$ with B on $\overline{QR}$ and C on $\overline{RS}$. Find the maximum possible area of $AQRS$.

Solution:

Let $\beta = \angle BAQ$ and $\gamma = \angle CAS$, so $\beta + \gamma = 90^\circ - A$. From the right triangles AQB and ASC, the sides of the rectangle are $AQ = 40\cos\beta$ and $AS = 31\cos\gamma$, so its area is

$$40 \cdot 31 \cos\beta\cos\gamma = 620\big(\cos(\beta - \gamma) + \cos(\beta + \gamma)\big) = 620\big(\cos(\beta - \gamma) + \sin A\big),$$

using the product-to-sum identity and $\cos(90^\circ - A) = \sin A$.

This is maximized when $\beta = \gamma$, which the constraint allows, giving area $620\left(1 + \frac{1}{5}\right) = 744$.

10. A strictly increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that for every positive integer k, the subsequence $a_{2k-1}, a_{2k}, a_{2k+1}$ is geometric and the subsequence $a_{2k}, a_{2k+1}, a_{2k+2}$ is arithmetic. Suppose that $a_{13} = 2016$. Find a_1.

Solution:

Write the common ratio of a_1, a_2, a_3 as $\frac{b}{a}$ in lowest terms, with $b > a \geq 1$ since the sequence increases. Because $a_3 = a_1 \left(\frac{b}{a}\right)^2$ is an integer and $\gcd(a, b) = 1$, we get $a^2 \mid a_1$; set $c = \frac{a_1}{a^2}$. Then $a_1 = ca^2$, $a_2 = cab$, $a_3 = cb^2$, and the arithmetic condition gives $a_4 = 2cb^2 - cab = cb(2b - a)$. Continuing, induction shows for every k that

$$a_{2k+1} = c\left(kb - (k-1)a\right)^2, \qquad a_{2k+2} = c\left(kb - (k-1)a\right)\left((k+1)b - ka\right).$$

In particular $a_{13} = c\,(6b - 5a)^2 = 2016 = 2^5 \cdot 3^2 \cdot 7$. Let $N = 6b - 5a$; then $N^2 \mid 2016$, so $N \leq 12$. But $N = a + 6(b - a) \geq a + 6 \geq 7$, and the only value in range with $N^2 \mid 2016$ is $N = 12$, giving $c = \frac{2016}{144} = 14$. From $6(b - a) = 12 - a$ we need $6 \mid a$ with $a \leq 6$, so $a = 6$ and $b = 7$, which are coprime.

Therefore $a_1 = ca^2 = 14 \cdot 36 = 504$. (Indeed the sequence begins $504, 588, 686, 784, 896, \ldots$ and reaches $a_{13} = 14 \cdot 12^2 = 2016$.)

11. Let $P(x)$ be a nonzero polynomial such that $(x - 1)P(x + 1) = (x + 2)P(x)$ for every real x, and $\left(P(2)\right)^2 = P(3)$. Then $P\left(\frac{7}{2}\right) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Setting $x = 1$ in the identity gives $0 = 3P(1)$, so $P(1) = 0$. Setting $x = 0$ gives $-P(1) = 2P(0)$, so $P(0) = 0$, and setting $x = -2$ gives $-3P(-1) = 0$, so $P(-1) = 0$. Hence $P(x) = x(x - 1)(x + 1)L(x)$ for some polynomial L.

Substituting back, $(x - 1)\,(x + 1)x(x + 2)L(x + 1) = (x + 2)\,x(x - 1)(x + 1)L(x)$, so $L(x + 1) = L(x)$ for all real x, which forces L to be a constant c. The normalization $\left(P(2)\right)^2 = P(3)$ reads $(6c)^2 = 24c$, so $c = \frac{2}{3}$.

Then

$$P\left(\tfrac{7}{2}\right) = \frac{2}{3} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{9}{2} = \frac{105}{4},$$

and $m + n = 105 + 4 = 109$.

12. Find the least positive integer m such that $m^2 - m + 11$ is a product of at least four not necessarily distinct primes.

Solution:

Let $e(m) = m^2 - m + 11$. Since $m^2 - m$ is always even, $e(m)$ is odd. Checking all residues shows $m^2 - m + 11$ is never 0 modulo $3, 5,$ or 7 either, so every prime factor of $e(m)$ is at least 11. A product of four such primes is at least $11^4 = 14641$, and the two smallest candidates are 11^4 and $11^3 \cdot 13 = 17303$.

For $e(m) = 14641$, the discriminant of $m^2 - m - 14630 = 0$ is 58521, which lies strictly between $241^2 = 58081$ and $242^2 = 58564$, so there is no integer solution. For $e(m) = 17303$: since $e(m) = m(m-1) + 11$ must be divisible by 11, either $m = 11k$ or $m = 11k + 1$. Trying $m = 11k$ gives $11k^2 - k + 1 = 1573$, that is $k(11k - 1) = 1572$, which $k = 12$ satisfies: $12 \cdot 131 = 1572$.

Since e is increasing for $m \geq 1$, every smaller m has $e(m) < 17303$, and the only four-prime value below that, 11^4, is unattainable. Hence the least m is $11 \cdot 12 = 132$, where $e(132) = 17303 = 11^3 \cdot 13$.

13. Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line $y = 24$. A fence is located at the horizontal line $y = 0$. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a point where $y = 0$, with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

Solution:

Horizontal jumps change nothing that matters, so let $T(y)$ be the expected number of jumps to reach the river from height y. Then $T(24) = 0$; for $1 \leq y \leq 23$ each jump goes up, down, or sideways with probabilities $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$, so

$$T(y) = 1 + \tfrac{1}{4}T(y+1) + \tfrac{1}{4}T(y-1) + \tfrac{1}{2}T(y),$$

which simplifies to $2T(y) = 4 + T(y-1) + T(y+1)$. At the fence the three equally likely moves give $T(0) = 1 + \frac{2}{3}T(0) + \frac{1}{3}T(1)$, that is $T(0) = 3 + T(1)$.

Summing $2T(y) = 4 + T(y-1) + T(y+1)$ over $y = 1, \ldots, 23$ telescopes to $T(1) + T(23) = 92 + T(0) + T(24)$. Substituting $T(0) = 3 + T(1)$ and $T(24) = 0$ yields $T(23) = 95$.

Now run the recurrence downward as $T(y-1) = 2T(y) - T(y+1) - 4$: from $T(24) = 0$ and $T(23) = 95$, we get $T(22) = 2 \cdot 95 - 0 - 4 = 186$ and $T(21) = 2 \cdot 186 - 95 - 4 = 273$. Freddy starts at height 21, so the answer is 273.

14. Centered at each lattice point in the coordinate plane are a circle radius $\frac{1}{10}$ and a square with sides of length $\frac{1}{5}$ whose sides are parallel to the coordinate axes. The line segment from $(0,0)$ to $(1001,429)$ intersects m of the squares and n of the circles. Find $m+n$.

Solution:

Since $\gcd(1001,429)=143$, the segment passes through the lattice points $(7k,3k)$ for $k=0,\ldots,143$ and consists of 143 translated copies of the segment from $(0,0)$ to $(7,3)$. The line is $y=\frac{3}{7}x$. It meets the square centered at (m,n) exactly when its height passes within $\frac{1}{10}$ of n for some x within $\frac{1}{10}$ of m, that is when $\left|\frac{3m}{7}-n\right| \le \frac{1}{10}+\frac{3}{7}\cdot\frac{1}{10}=\frac{1}{7}$, or equivalently $|3m-7n|\le 1$.

For $0\le m\le 7$ the solutions are $(0,0)$ and $(7,3)$ with $3m-7n=0$, and $(2,1)$ and $(5,2)$ with $3m-7n=\mp 1$. In the first two the line passes through the center, so it meets the circle as well. In the other two, equality means the line passes exactly through a corner of the square (for $(2,1)$, the corner $(2.1,0.9)$), while its distance to the center is $\frac{1}{\sqrt{3^2+7^2}}=\frac{1}{\sqrt{58}}>\frac{1}{10}$, so it misses the circle. Thus each copy of the segment meets 4 squares and 2 circles.

The 142 interior lattice points $(7k,3k)$ are each shared by two consecutive copies, so $m=4\cdot 143-142=430$ and $n=2\cdot 143-142=144$, giving $m+n=574$.

15. Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

Solution:

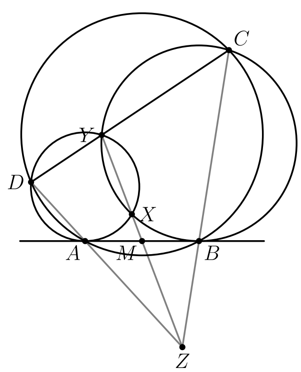
Line AD is the radical axis of ω and ω_1, line BC that of ω and ω_2, and line XY that of ω_1 and ω_2, so the three lines meet at the radical center Z. (They cannot be parallel: that would force a symmetric configuration with $XC = XD$.) Let $M = XY \cap AB$. The power of M with respect to each circle gives $MA^2 = MX \cdot MY = MB^2$, so M is the midpoint of $\overline{AB}$, with X between M and Y.

Since $ADYX$ is cyclic, $\angle XAZ = \angle XYD$, and since $BCYX$ is cyclic, $\angle XBZ = \angle XYC$; as C, Y, D are collinear these add to 180°, so $ZAXB$ is cyclic. The tangent-chord angle at B gives $\angle XYB = \angle ABX = \angle AZX$, so $BY \parallel ZA$, and symmetrically $AY \parallel ZB$. Hence $AYBZ$ is a parallelogram, and since M is the midpoint of diagonal $\overline{AB}$, it is also the midpoint of $\overline{ZY}$: therefore $XZ = XM + MZ = MX + MY$. Moreover $\angle XCZ = \angle XYB = \angle XZD$ and (by the tangent-chord angle at A) $\angle XZC = \angle XAB = \angle XYA = \angle XDZ$, so triangles XZC and XDZ are similar, giving $XZ^2 = XC \cdot XD$.

Putting it together,

$$AB^2 = 4MA^2 = 4\,MX \cdot MY = (MX + MY)^2 - (MY - MX)^2 = XZ^2 - XY^2 = XC \cdot XD - XY^2,$$

which equals $67 \cdot 37 - 47^2 = 2479 - 2209 = 270$.



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