2015 AIME II Solutions

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1. Let N be the least positive integer that is both 22 percent less than one integer and 16 percent greater than another integer. Find the remainder when N is divided by $1000.$

Solution:

The conditions say $N = \frac{78}{100}a = \frac{39}{50}a$ and $N = \frac{116}{100}b = \frac{29}{25}b$ for some integers a and $b.$ Since $\gcd(39, 50) = 1,$ the first equation forces $50 \mid a,$ so N is a multiple of $39;$ since $\gcd(29, 25) = 1,$ the second forces $25 \mid b,$ so N is a multiple of $29.$

The least positive integer divisible by both is $N = 39 \cdot 29 = 1131,$ achieved with $a = 1450$ and $b = 975.$ The remainder upon division by 1000 is $131.$

2. In a new school 40 percent of the students are freshmen, 30 percent are sophomores, 20 percent are juniors, and 10 percent are seniors. All freshmen are required to take Latin, and 80 percent of the sophomores, 50 percent of the juniors, and 20 percent of the seniors elect to take Latin. The probability that a randomly chosen Latin student is a sophomore is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Assume the school has 100 students. The Latin students are then 40 freshmen, $30(0.8) = 24$ sophomores, $20(0.5) = 10$ juniors, and $10(0.2) = 2$ seniors, for a total of 76.

The probability that a random Latin student is a sophomore is $\frac{24}{76} = \frac{6}{19}$, so $m + n = 6 + 19 = 25$.

3. Let m be the least positive integer divisible by 17 whose digits sum to 17. Find m.

Solution:

Every number is congruent to its digit sum modulo 9, so $m = 17n$ must satisfy $17n \equiv 17 \pmod 9$, that is $8n \equiv 8 \pmod 9$, which gives $n \equiv 1 \pmod 9$.

Checking the candidates in increasing order: $n = 1, 10, 19$ give $17, 170, 323$, with digit sums 8 each, but $n = 28$ gives $17 \cdot 28 = 476$ with digit sum $4 + 7 + 6 = 17$. So $m = 476$.

4. In an isosceles trapezoid, the parallel bases have lengths $\log 3$ and $\log 192$, and the altitude to these bases has length $\log 16$. The perimeter of the trapezoid can be written in the form $\log 2^p 3^q$, where p and q are positive integers. Find $p + q$.

Solution:

Dropping altitudes from the ends of the short base, each leg is the hypotenuse of a right triangle whose legs are the altitude $\log 16 = 4 \log 2$ and half the difference of the bases, $\frac{1}{2}(\log 192 - \log 3) = \frac{1}{2} \log 64 = 3 \log 2$. By the 3-4-5 ratio, each leg has length $5 \log 2$.

The perimeter is

$$\log 3 + \log 192 + 2 \cdot 5 \log 2 = \log(3 \cdot 192) + \log 2^{10} = \log(2^6 3^2) + \log 2^{10} = \log 2^{16} 3^2,$$

so $p + q = 16 + 2 = 18$.

5. Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the least positive integer n such that the probability that the two selected squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

Solution:

Each of the n rows contains $n - 1$ horizontally adjacent pairs, so there are $n(n-1)$ horizontal pairs and likewise $n(n-1)$ vertical pairs. Out of $\binom{n^2}{2} = \frac{n^2(n^2-1)}{2}$ equally likely pairs, the probability of adjacency is

$$\frac{2n(n-1) \cdot 2}{n^2(n^2-1)} = \frac{4}{n(n+1)}.$$

We need $n(n+1) > 4 \cdot 2015 = 8060$. Since $89 \cdot 90 = 8010$ and $90 \cdot 91 = 8190$, the least such n is 90.

6. Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$ for some positive integers a and c. Can you tell me the values of a and c?"

After some calculations, Jon says, "There is more than one such polynomial."

Steve says, "You're right. Here is the value of a." He writes down a positive integer and asks, "Can you tell me the value of c?"

Jon says, "There are still two possible values of c."

Find the sum of the two possible values of c.

Solution:

Dividing by 2, the roots $r \leq s \leq t$ satisfy $r + s + t = a, rs + rt + st = \frac{a^2-81}{2}$, and $rst = \frac{c}{2}$. Therefore

$$r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + rt + st) = a^2 - (a^2 - 81) = 81.$$

The triples of positive integers whose squares sum to 81 are $(1, 4, 8)$, $(4, 4, 7)$, and $(3, 6, 6)$, with $a = r + s + t$ equal to $13, 15$, and 15. Since knowing a still left Jon two choices, $a = 15$, and the two polynomials come from $(4, 4, 7)$ and $(3, 6, 6)$.

The corresponding values of $c = 2rst$ are $2 \cdot 4 \cdot 4 \cdot 7 = 224$ and $2 \cdot 3 \cdot 6 \cdot 6 = 216$, with sum $224 + 216 = 440$.

7. Triangle ABC has side lengths $AB = 12, BC = 25$, and $CA = 17$. Rectangle $PQRS$ has vertex P on $\overline{AB}$, vertex Q on $\overline{AC}$, and vertices R and S on $\overline{BC}$. In terms of the side length $PQ = w$, the area of $PQRS$ can be expressed as the quadratic polynomial

$$\text{Area}(PQRS) = \alpha w - \beta \cdot w^2.$$

Then the coefficient $\beta = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

By Heron's formula with $s = 27$, the area of ABC is $\sqrt{27 \cdot 2 \cdot 10 \cdot 15} = \sqrt{8100} = 90$, so the altitude from A to $\overline{BC}$ has length $h = \frac{2 \cdot 90}{25} = \frac{36}{5}$.

Since $\overline{PQ} \parallel \overline{BC}$, triangle APQ is similar to ABC with ratio $\frac{w}{25}$, so the distance from A down to line PQ is $\frac{w}{25}h$, and the rectangle's height is $PS = h - \frac{w}{25}h$. The area is

$$w \cdot h\left(1 - \frac{w}{25}\right) = \frac{36}{5}\,w - \frac{36}{125}\,w^2.$$

Thus $\beta = \frac{36}{125}$, and $m + n = 36 + 125 = 161$.

8. Let a and b be positive integers satisfying $\frac{ab+1}{a+b} < \frac{3}{2}$. The maximum possible value of $\frac{a^3b^3+1}{a^3+b^3}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

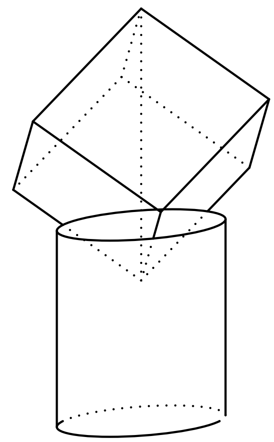
If $a = 1$ or $b = 1$, then $\frac{a^3b^3+1}{a^3+b^3} = 1$. So assume $a, b \geq 2$. Clearing denominators, the hypothesis says $2ab + 2 < 3a + 3b$, and multiplying by 2 and rearranging gives

$$(2a-3)(2b-3) = 4ab - 6a - 6b + 9 < 5.$$

For $a, b \geq 2$ both factors are positive odd integers, so up to symmetry the only options are $(a, b) = (2, 2)$ and $(2, 3)$ (both of which do satisfy the original inequality, while $(3, 3)$ gives the product 9).

The values are $\frac{65}{16}$ for $(2, 2)$ and $\frac{8 \cdot 27+1}{8+27} = \frac{217}{35} = \frac{31}{5}$ for $(2, 3)$. The larger is $\frac{31}{5}$, so $p + q = 31 + 5 = 36$.

9. A cylindrical barrel with radius 4 feet and height 10 feet is full of water. A solid cube with side length 8 feet is set into the barrel so that the diagonal of the cube is vertical. The volume of water thus displaced is v cubic feet. Find v^2.



Solution:

The displaced volume equals the volume of the part of the cube lying below the plane of the barrel's rim. By symmetry that region is a tetrahedron cut from the bottom corner of the cube: three mutually perpendicular edges of equal length ℓ along the cube's edges, capped by an equilateral triangle in the rim plane. The equilateral cross-section is inscribed in the rim circle of radius 4, so its side length is $4\sqrt{3}$, and therefore $\ell = \frac{4\sqrt{3}}{\sqrt{2}} = 2\sqrt{6}$.

Taking one of the right isosceles faces as the base, the volume is

$$\frac{1}{3}\left(\frac{1}{2}\ell^2\right)\ell = \frac{\ell^3}{6} = \frac{(2\sqrt{6})^3}{6} = \frac{48\sqrt{6}}{6} = 8\sqrt{6}.$$

Thus $v = 8\sqrt{6}$ and $v^2 = 64 \cdot 6 = 384$.

10. Call a permutation $a_1, a_2, \ldots, a_n$ of the integers $1, 2, \ldots, n$ *quasi-increasing* if $a_k \leq a_{k+1} + 2$ for each $1 \leq k \leq n-1$. For example, 53421 and 14253 are quasi-increasing permutations of the integers $1, 2, 3, 4, 5$, but 45123 is not. Find the number of quasi-increasing permutations of the integers $1, 2, \ldots, 7$.

Solution:

Let S_n be the number of quasi-increasing permutations of $1, \ldots, n$. Insert n into a quasi-increasing permutation of $1, \ldots, n-1$: the entry following n must be at least $n-2$, so n can go immediately before $n-1$, immediately before $n-2$, or at the very end — exactly 3 positions, and each insertion keeps every other adjacent condition intact.

Conversely, deleting n from a quasi-increasing permutation of $1, \ldots, n$ leaves a quasi-increasing permutation of $1, \ldots, n-1$, since the entries around the deleted n satisfy $a_{k-1} \leq n-1 \leq a_{k+1} + 2$ when $n \geq 3$. So $S_n = 3S_{n-1}$ for $n \geq 3$.

Since $S_2 = 2$, we get $S_7 = 2 \cdot 3^5 = 486$.

11. The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

The central angle over $\overline{BC}$ is $\angle BOC = 2\angle A$, and $OB = OC$ makes triangle OBC isosceles, so $\angle OBC = 90^\circ - \angle A$. In triangle OBQ the angle at O is 90°, hence

$$\angle BQP = 90^\circ - \angle OBQ = 90^\circ - (90^\circ - \angle A) = \angle A.$$

Triangles BQP and BAC share the angle at B and have $\angle BQP = \angle BAC$, so they are similar, giving $\frac{BP}{BC} = \frac{BQ}{BA}$. Therefore

$$BP = \frac{BQ \cdot BC}{BA} = \frac{4.5 \cdot 4}{5} = \frac{18}{5},$$

and $m + n = 18 + 5 = 23.$

12. There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

Solution:

The condition says every maximal run of identical letters has length at most 3. Let s_n count the valid strings of length n whose first letter is A; by symmetry the answer is $2s_{10}$. Removing the first run (of length $1, 2,$ or 3) leaves a valid shorter string beginning with B, so

$$s_n = s_{n-1} + s_{n-2} + s_{n-3}.$$

Starting from $s_1 = 1, s_2 = 2, s_3 = 4$, the sequence runs $7, 13, 24, 44, 81, 149, 274$, so $s_{10} = 274.$

The number of valid strings is $2 \cdot 274 = 548.$

13. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_n = \sum_{k=1}^{n} \sin(k)$, where k represents radian measure. Find the index of the 100th term for which $a_n < 0$.

Solution:

Multiplying each term by $2\sin\frac{1}{2}$ and using $2\sin k \sin\frac{1}{2} = \cos\left(k - \frac{1}{2}\right) - \cos\left(k + \frac{1}{2}\right)$, the sum telescopes:

$$a_n = \frac{\cos\frac{1}{2} - \cos\left(n + \frac{1}{2}\right)}{2\sin\frac{1}{2}}.$$

So $a_n < 0$ exactly when $\cos\left(n + \frac{1}{2}\right) > \cos\frac{1}{2}$, which happens exactly when $n + \frac{1}{2}$ is within $\frac{1}{2}$ of a multiple of 2π :

$$2\pi m - \tfrac{1}{2} < n + \tfrac{1}{2} < 2\pi m + \tfrac{1}{2}, \qquad \text{i.e.} \qquad 2\pi m - 1 < n < 2\pi m.$$

Each interval $(2\pi m - 1,\ 2\pi m)$ has length 1 and contains exactly one integer, namely $\lfloor 2\pi m \rfloor$.

Hence the 100th negative term has index $\lfloor 200\pi \rfloor$. Since $3.14 < \pi < 3.145$, we have $628 < 200\pi < 629$, so the index is 628.

14. Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Solution:

The equations factor as $x^4y^4(x+y) = 810$ and $x^3y^3(x^3+y^3) = 945$. With $s = x+y$ and $p = xy$, using $x^3+y^3 = s(s^2-3p)$, they become $p^4s = 810$ and $p^3s\left(s^2-3p\right) = 945$. Dividing,

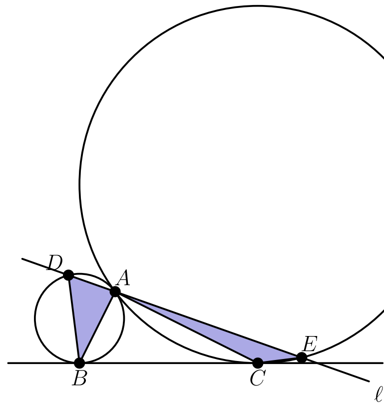
$$\frac{s^2-3p}{p} = \frac{945}{810} = \frac{7}{6}, \qquad \text{so} \qquad 6s^2 = 25p.$$

Substituting $p = \frac{6s^2}{25}$ into $p^4s = 810$ gives $\left(\frac{6}{25}\right)^4 s^9 = 810$, so $s^9 = 810 \cdot \frac{390625}{1296} = \frac{1953125}{8}$, which means $s^3 = \frac{125}{2}$. Then $ps = \frac{6s^3}{25} = 15$ and $p^3 = \frac{216s^6}{25^3} = \frac{216 \cdot 15625/4}{15625} = 54$.

Finally

$$2x^3 + (xy)^3 + 2y^3 = 2(s^3 - 3ps) + p^3 = 2\left(\frac{125}{2} - 45\right) + 54 = 35 + 54 = 89.$$

15. Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ so that line BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ, and the areas of $\triangle DBA$ and $\triangle ACE$ are equal. This common area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution:

Place line BC on the x-axis, so the centers are $P=(0,1)$ and $Q=(4,4)$ (their distance is $1+4=5$), with $B=(0,0)$, $C=(4,0)$, and the tangency point $A=P+\frac{1}{5}(Q-P)=\left(\frac{4}{5},\frac{8}{5}\right)$. The homothety centered at A with ratio -4 carries $\mathcal{P}$ to $\mathcal{Q}$ and D to E, so $AE=4\,AD$. Since $[DBA]=\frac{1}{2}AD\cdot d(B,\ell)$ and $[ACE]=\frac{1}{2}AE\cdot d(C,\ell)$, the equal-area condition is $d(B,\ell)=4\,d(C,\ell)$ with B and C on the same side of ℓ.

Write ℓ as $u\left(x-\frac{4}{5}\right)+v\left(y-\frac{8}{5}\right)=0$. Its signed values at B and C are $-\frac{4u+8v}{5}$ and $\frac{16u-8v}{5}$, so the same-side ratio-4 condition reads $-(4u+8v)=4(16u-8v)$, giving $24v=68u$, i.e. $v=\frac{17}{6}u$. Taking $(u,v)=(6,17)$, the line is $6x+17y=32$.

Then $d(B,\ell)=\frac{32}{\sqrt{325}}$, and the center $P=(0,1)$ is at distance $\frac{15}{\sqrt{325}}$ from ℓ, so the chord gives $AD=2\sqrt{1-\frac{225}{325}}=\frac{20}{\sqrt{325}}$. The common area is

$$\frac{1}{2}\cdot\frac{20}{\sqrt{325}}\cdot\frac{32}{\sqrt{325}}=\frac{320}{325}=\frac{64}{65},$$

so $m+n=64+65=129$.

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