2015 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
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1. The expressions $A = 1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + 37 \times 38 + 39$ and $B = 1 + 2 \times 3 + 4 \times 5 + \cdots + 36 \times 37 + 38 \times 39$ are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between integers A and B.

Solution:

Subtract term by term:

$$B - A = (1 - 39) + (2 \times 3 - 1 \times 2) + (4 \times 5 - 3 \times 4) + \cdots + (38 \times 39 - 37 \times 38).$$

Each parenthesized difference has the form $(2k+1)(2k) - (2k-1)(2k) = 4k$ for $k = 1, 2, \ldots, 19.$

Therefore $B - A = -38 + 4(1 + 2 + \cdots + 19) = -38 + 4 \cdot 190 = 722.$

2. The nine delegates to the Economic Cooperation Conference include 2 officials from Mexico, 3 officials from Canada, and 4 officials from the United States. During the opening session, three of the delegates fall asleep. Assuming that the three sleepers were determined randomly, the probability that exactly two of the sleepers are from the same country is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

There are $\binom{9}{3}=84$ equally likely sets of three sleepers. Exactly two sleepers come from the same country when one country supplies exactly two of them and the third sleeper comes from a different country: $\binom{4}{2}(2+3)=30$ ways with the pair from the United States, $\binom{3}{2}(2+4)=18$ with the pair from Canada, and $\binom{2}{2}(3+4)=7$ with the pair from Mexico.

The probability is $\frac{30+18+7}{84}=\frac{55}{84}$, already in lowest terms, so $m+n=55+84=139$.

3. There is a prime number p such that $16p+1$ is the cube of a positive integer. Find p.

Solution:

Write $16p+1=n^3$, so $16p=n^3-1=(n-1)(n^2+n+1)$. Since $16p+1$ is odd, n is odd, and n^2+n+1 is odd as well. Therefore all four factors of 2 must divide $n-1:$ write $n-1=16k$, which gives $p=k(n^2+n+1)$. For p to be prime we need $k=1$, so $n=17$.

Then $p=17^2+17+1=307$, which is indeed prime, and $16\cdot 307+1=4913=17^3$.

4. Point B lies on line segment $\overline{AC}$ with $AB = 16$ and $BC = 4$. Points D and E lie on the same side of line AC forming equilateral triangles $\triangle ABD$ and $\triangle BCE$. Let M be the midpoint of $\overline{AE}$, and N be the midpoint of $\overline{CD}$. The area of $\triangle BMN$ is x. Find x^2.

Solution:

Place $B = (0, 0)$, $A = (-16, 0)$, and $C = (4, 0)$. Each equilateral triangle has its apex above the midpoint of its base at height $\frac{\sqrt{3}}{2}$ times the side, so $D = (-8, 8\sqrt{3})$ and $E = (2, 2\sqrt{3})$. The midpoints are $M = (-7, \sqrt{3})$ and $N = (-2, 4\sqrt{3})$.

Now $BM^2 = 49 + 3 = 52$, $BN^2 = 4 + 48 = 52$, and $MN^2 = 25 + 27 = 52$, so $\triangle BMN$ is equilateral with side $\sqrt{52}$. Its area is $x = \frac{\sqrt{3}}{4} \cdot 52 = 13\sqrt{3}$, so $x^2 = 169 \cdot 3 = 507$.

5. In a drawer Sandy has 5 pairs of socks, each pair a different color. On Monday Sandy selects two individual socks at random from the 10 socks in the drawer. On Tuesday Sandy selects 2 of the remaining 8 socks at random and on Wednesday two of the remaining 6 socks at random. The probability that Wednesday is the first day Sandy selects matching socks is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

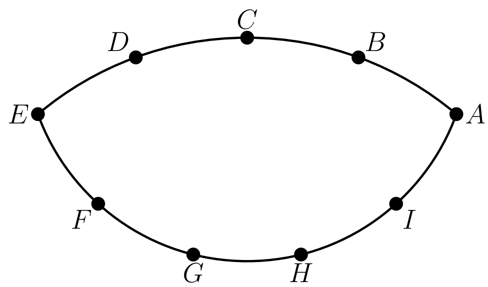
Solution:

Imagine dealing all ten socks out two per day for five days; every assignment of unordered pairs to days is equally likely, and permuting the days does not change this distribution. Swapping Monday and Wednesday therefore shows that the desired probability (mismatch, mismatch, match) equals the probability of a match on Monday followed by mismatches on Tuesday and Wednesday.

That pattern is easy to compute in order. Monday matches with probability $\frac{1}{9}$ (the second sock must be the first sock's mate). The remaining 8 socks then form 4 complete pairs, so Tuesday mismatches with probability $1-\frac{4}{\binom{8}{2}}=\frac{6}{7}$. Tuesday's mismatch breaks two pairs, leaving 2 complete pairs among the 6 remaining socks, so Wednesday mismatches with probability $1-\frac{2}{\binom{6}{2}}=\frac{13}{15}$.

The probability is $\frac{1}{9}\cdot\frac{6}{7}\cdot\frac{13}{15}=\frac{26}{315}$, so $m+n=26+315=341.$

6. Points A, B, C, D, and E are equally spaced on a minor arc of a circle. Points E, F, G, H, I, and A are equally spaced on a minor arc of a second circle with center C as shown in the figure below. The angle $\angle ABD$ exceeds $\angle AHG$ by 12°. Find the degree measure of $\angle BAG$.



Solution:

Let $\alpha = \angle ECF = \angle FCG = \angle GCH = \angle HCI = \angle ICA$, the common central angle of the second circle, so $\angle ACE = 5\alpha$. Since C also lies on the first circle, $\angle ACE$ is an inscribed angle there, so the arc AE not containing C measures 10α, and each of the four equal arcs AB, BC, CD, DE measures $\frac{360^{\circ}-10\alpha}{4} = 90^{\circ} - \frac{5\alpha}{2}$.

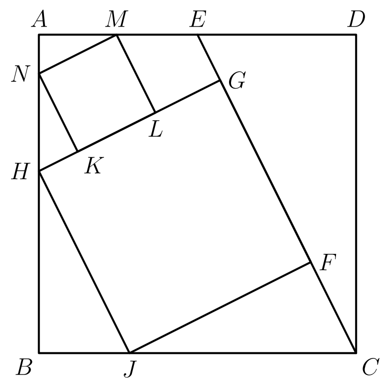
Angle ABD subtends the arc AD not containing B, which is $360^{\circ} - 3\left(90^{\circ} - \frac{5\alpha}{2}\right)$, so $\angle ABD = 45^{\circ} + \frac{15\alpha}{4}$. Angle AHG subtends the second circle's arc AG not containing H, which is $360^{\circ} - 3\alpha$, so $\angle AHG = 180^{\circ} - \frac{3\alpha}{2}$. The given condition reads

$$\left(45^{\circ} + \frac{15\alpha}{4}\right) - \left(180^{\circ} - \frac{3\alpha}{2}\right) = \frac{21\alpha}{4} - 135^{\circ} = 12^{\circ},$$

so $\alpha = 28^{\circ}$.

Finally, $\angle BAE$ subtends the first circle's arc $BCDE = 3\left(90^{\circ} - \frac{5\alpha}{2}\right) = 60^{\circ}$, giving $\angle BAE = 30^{\circ}$, and $\angle EAG$ subtends the second circle's arc $EFG = 2\alpha$, giving $\angle EAG = 28^{\circ}$. Hence $\angle BAG = \angle BAE + \angle EAG = 30^{\circ} + 28^{\circ} = 58^{\circ}$.

7. In the diagram below, $ABCD$ is a square. Point E is the midpoint of $\overline{AD}$. Points F and G lie on $\overline{CE}$, and H and J lie on $\overline{AB}$ and $\overline{BC}$, respectively, so that $FGHJ$ is a square. Points K and L lie on $\overline{GH}$, and M and N lie on $\overline{AD}$ and $\overline{AB}$, respectively, so that $KLMN$ is a square. The area of $KLMN$ is 99. Find the area of $FGHJ$.



Solution:

Let $AE = s$, so the big square has side $2s$ and $CE = s\sqrt{5}$. The right triangles CDE, JFC, HBJ, NKH, and MAN are all similar, with legs in ratio $1:2$. Let x be the side of $FGHJ$. In $\triangle HBJ$ the hypotenuse is $HJ = x$, so $BJ = \frac{x}{\sqrt{5}}$ and $HB = \frac{2x}{\sqrt{5}}$; in $\triangle JFC$ the longer leg is $JF = x$, so the hypotenuse is $JC = \frac{x\sqrt{5}}{2}$. Then

$$2s = BC = BJ + JC = x\left(\frac{1}{\sqrt{5}} + \frac{\sqrt{5}}{2}\right) = \frac{7x}{2\sqrt{5}},$$

so $x = \frac{4\sqrt{5}\,s}{7}$.

Next, $AH = 2s - HB = 2s - \frac{8s}{7} = \frac{6s}{7}$. The identical decomposition along $\overline{AB}$ for the square $KLMN$ of side y gives $\frac{6s}{7} = AH = AN + NH = y\left(\frac{1}{\sqrt{5}} + \frac{\sqrt{5}}{2}\right)$. Dividing the two equations, $\frac{x}{y} = \frac{2s}{6s/7} = \frac{7}{3}$.

The areas are therefore in ratio $\left(\frac{7}{3}\right)^2 = \frac{49}{9}$, so the area of $FGHJ$ is $99 \cdot \frac{49}{9} = 539.$

8. For positive integer n, let $s(n)$ denote the sum of the digits of n. Find the smallest positive integer n satisfying $s(n) = s(n + 864) = 20$.

Solution:

Each carry in an addition replaces 10 in one place by 1 in the next, lowering the digit sum by 9. Hence $s(n + 864) = s(n) + s(864) - 9c = 20 + 18 - 9c$, where c is the number of carries, and $s(n + 864) = 20$ forces $c = 2$. For a three-digit candidate n with digits t, u, v summing to 20 : since $u + v \leq 18$, we have $t \geq 2$, so the hundreds place always carries ($t + 8 \geq 10$), and exactly one of the units and tens places carries.

If the units carry and the tens do not, the tens computation $u + 6 + 1$ must stay below 10, so $u \leq 2$; then $t = 20 - u - v \geq 20 - 2 - 9 = 9$, forcing $n = 929$. If the tens carry and the units do not, then $v + 4 \leq 9$ gives $v \leq 5$, so $t = 20 - u - v \geq 20 - 9 - 5 = 6$, and $t = 6, u = 9, v = 5$ works: $n = 695$.

Indeed $s(695) = 20$ and $695 + 864 = 1559$ with $s(1559) = 20$, so the smallest such n is 695.

9. Let S be the set of all ordered triples of integers (a_1, a_2, a_3) with $1 \leq a_1, a_2, a_3 \leq 10$. Each ordered triple in S generates a sequence according to the rule $a_n = a_{n-1} \cdot |a_{n-2} - a_{n-3}|$ for $n \geq 4$. Find the number of such sequences for which $a_n = 0$ for some n.

Solution:

If $a_{k-1} = a_k$ then $a_{k+2} = a_{k+1}|a_k - a_{k-1}| = 0$, and if $|a_k - a_{k-1}| = 1$ then $a_{k+2} = a_{k+1}$, so $a_{k+4} = 0$. Hence every triple of one of the forms $(j, j, k), (j, k, k), (j, j \pm 1, k), (j, k, k \pm 1)$ produces a 0. These forms contain $100 + 100 + 4 \cdot 90 = 560$ triples, but triples fitting two forms are counted twice: the 10 of the form (j, j, j), the 9 in each of the six families $(j, j, j \pm 1), (j, j \pm 1, j), (j, j \pm 1, j \pm 1)$ (matching signs), and the 8 in each of $(j, j+1, j+2)$ and $(j, j-1, j-2)$. That leaves $560 - 10 - 54 - 16 = 480$ triples.

A few other triples also work: if $(a_1, a_2, a_3) = (j, j \pm 2, 1)$, then $a_4 = 2$ and $|a_4 - a_3| = 1$, so $a_8 = 0$. These 16 triples include $(3, 1, 1)$ and $(4, 2, 1)$, which were already counted, so they add 14 new ones, for $480 + 14 = 494$.

No other triple reaches 0 : if both consecutive differences are at least 2 and $a_3 \geq 2$, then $a_4 = a_3|a_2 - a_1| \geq 2a_3 > a_3$ and $|a_4 - a_3| \geq a_3 \geq 2$, so inductively the terms grow forever and no factor ever vanishes. If instead $a_3 = 1$ with $|a_2 - a_1| \geq 3$, then $a_4 \geq 3$ and $|a_4 - a_3| \geq 2$, and the same growth takes over. The count is 494.

10. Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

Solution:

Each of $f(x) - 12$ and $f(x) + 12$ is a cubic, so each vanishes at exactly three of $1, 2, 3, 5, 6, 7$. Writing them as $c(x - r_1)(x - r_2)(x - r_3)$ and $c(x - s_1)(x - s_2)(x - s_3)$, the two cubics differ by the constant 24, so their x^2 and x coefficients agree: the root triples have equal sums and equal sums of pairwise products. The only partition of $\{1, 2, 3, 5, 6, 7\}$ into two triples of equal sum is $\{2, 3, 7\}$ and $\{1, 5, 6\}$ (each summing to 12), and indeed both have pairwise-product sum 41.

Replacing f by $-f$ if necessary (which does not change $|f(0)|$), we have $f(x) = c(x - 2)(x - 3)(x - 7) + 12 = c(x - 1)(x - 5)(x - 6) - 12$. Setting $x = 0$ gives $-42c + 12 = -30c - 12$, so $c = 2$ and $f(0) = -42 \cdot 2 + 12 = -72$. Thus $|f(0)| = 72$.

11. Triangle ABC has positive integer side lengths with $AB = AC$. Let I be the intersection of the bisectors of $\angle B$ and $\angle C$. Suppose $BI = 8$. Find the smallest possible perimeter of $\triangle ABC$.

Solution:

Let M be the midpoint of $\overline{BC}$; by symmetry $A, I,$ and M are collinear with $AM \perp BC$. With $a = AB$ and $b = BM$, right triangles ABM and IBM give $\cos \angle ABM = \frac{b}{a}$ and $\cos \angle IBM = \frac{b}{8}$. Since BI bisects $\angle ABM$, the double-angle formula yields

$$\frac{b}{a} = 2\left(\frac{b}{8}\right)^2 - 1, \qquad \text{so} \qquad a = \frac{32b}{b^2 - 32}.$$

Writing $c = BC = 2b$, this becomes $a = \frac{64c}{c^2-128}$. We need $c^2 > 128$, so $c \geq 12$, while $\cos \angle IBM = \frac{b}{8} < 1$ forces $c < 16$. Testing $c = 12, 13, 14, 15$, only $c = 12$ makes a an integer, namely $a = \frac{768}{16} = 48$.

The triangle with sides $48, 48, 12$ satisfies all the conditions, and its perimeter is $48 + 48 + 12 = 108$.

12. Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:

A 1000-element subset has least element j exactly when it contains j together with 999 larger elements, so $\binom{2015-j}{999}$ of the subsets have least element j. The mean is therefore $\frac{\sum_j j\binom{2015-j}{999}}{\binom{2015}{1000}}$.

The numerator counts something concrete: to build a 1001-element subset of $\{0, 1, \ldots, 2015\}$ whose second-smallest element is j, choose its smallest element from $\{0, \ldots, j-1\}$ (j ways) and its top 999 elements from $\{j+1, \ldots, 2015\}$. Summing over j produces every 1001-element subset exactly once, so $\sum_j j\binom{2015-j}{999} = \binom{2016}{1001}$.

Hence the mean is $\frac{\binom{2016}{1001}}{\binom{2015}{1000}} = \frac{2016}{1001} = \frac{288}{143}$, which is in lowest terms, and $p + q = 288 + 143 = 431$.

13. With all angles measured in degrees, the product $\prod_{k=1}^{45} \csc^2(2k-1)^\circ = m^n$, where m and n are integers greater than 1. Find $m+n$.

Solution:

Let $P = \sin 1^\circ \sin 3^\circ \cdots \sin 89^\circ$ and $Q = \sin 2^\circ \sin 4^\circ \cdots \sin 88^\circ$, so the desired product is $\frac{1}{P^2}$. Then $PQ = \prod_{k=1}^{89} \sin k^\circ$, and multiplying this by itself in reverse order, using $\sin(90-k)^\circ = \cos k^\circ$, gives

$$P^2Q^2 = \prod_{k=1}^{89} \sin k^\circ \cos k^\circ.$$

Multiply by 2^{89} and use $2 \sin k^\circ \cos k^\circ = \sin 2k^\circ$:

$$2^{89}P^2Q^2 = \prod_{k=1}^{89} \sin 2k^\circ = \left(\prod_{k=1}^{44} \sin 2k^\circ\right)\left(\prod_{k=46}^{89} \sin 2k^\circ\right) = Q \cdot Q,$$

since $\sin 90^\circ = 1$ and $\sin(180-x)^\circ = \sin x^\circ$ turns the second half into Q as well.

Because $Q \neq 0$, it follows that $P^2 = 2^{-89}$, so $\prod_{k=1}^{45} \csc^2(2k-1)^\circ = 2^{89}$. Since 89 is prime, the only representation m^n with $m, n > 1$ is $m=2, n=89$, and $m+n=91$.

14. For each integer $n \geq 2$, let $A(n)$ be the area of the region in the coordinate plane defined by the inequalities $1 \leq x < n$ and $0 \leq y \leq x\lfloor\sqrt{x}\rfloor$, where $\lfloor\sqrt{x}\rfloor$ is the greatest integer not exceeding $\sqrt{x}$. Find the number of values of n with $2 \leq n \leq 1000$ for which $A(n)$ is an integer.

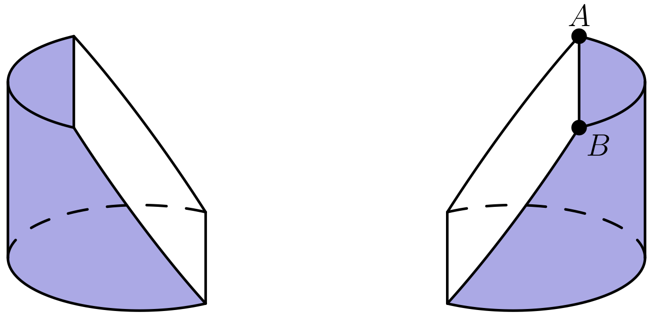
Solution:

On the strip $m \leq x < m+1$ we have $\lfloor\sqrt{x}\rfloor = k = \lfloor\sqrt{m}\rfloor$, so the region above it is a trapezoid under $y = kx$ with area $A(m+1) - A(m) = \frac{(2m+1)k}{2}$: an integer when k is even, a half-integer when k is odd. Hence as n grows by 1, the integrality of $A(n)$ is unchanged while k is even and flips at every step while k is odd.

Consider the block of $2k+1$ values $k^2 < n \leq (k+1)^2$. Starting from $A(1) = 0$, the statuses of $A(k^2)$ cycle with period 4 : integer for $k \equiv 0, 1$ and non-integer for $k \equiv 2, 3 \pmod 4$ (an odd block flips the status an odd number of times, an even block preserves it). Counting integer values of $A(n)$ inside each block: for $k = 4j-3$ the block alternates, beginning and ending with non-integers, giving $4j-3$; for $k = 4j-2$ every value is a non-integer, giving 0; for $k = 4j-1$ it alternates, beginning and ending with integers, giving $4j$; for $k = 4j$ all $8j+1$ values are integers.

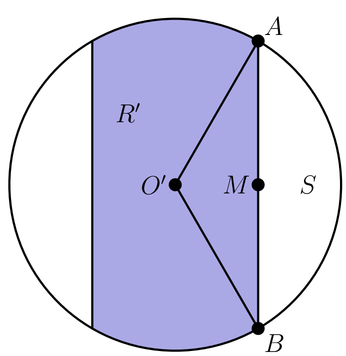
For $j = 1, \ldots, 7$, covering $2 \leq n \leq 29^2$, the four blocks contribute $(4j-3) + 0 + 4j + (8j+1) = 16j - 2$ integers, totaling $\sum_{j=1}^{7}(16j-2) = 434$. Then the block $k = 29$ contributes 29 integers for $841 < n \leq 900$, the block $k = 30$ contributes none, and for $k = 31$ the alternation over $961 < n \leq 1000$ begins with an integer at $n = 962$ and gives 20 more. The total is $434 + 29 + 20 = 483$.

15. A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge of one of the circular faces of the cylinder so that arc $\overset{\frown}{AB}$ on that face measures 120°. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of these unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.



Solution:

Stand the block on the face containing A and B, and let O' be the center of that face, M the midpoint of $\overline{AB}$, and O the center of the cylinder. The cutting plane meets the bottom face in chord $\overline{AB}$ and, by symmetry through O, meets the top face in the reflected chord, so the cut face projects vertically onto the region R' between chord $\overline{AB}$ and its mirror image through O' (shaded below). Each 120° circular segment cut off has area $\frac{1}{3}\pi \cdot 6^2 - \frac{1}{2} \cdot 6 \cdot 6 \sin 120^\circ = 12\pi - 9\sqrt{3}$, so R' has area $36\pi - 2\left(12\pi - 9\sqrt{3}\right) = 12\pi + 18\sqrt{3}$.



Since $\overset{\frown}{AB} = 120^\circ$, triangle $AO'B$ gives $O'M = 6\cos 60^\circ = 3$, and $OO' = 4$, so $OM = 5$. The cut face is planar and tilted from the horizontal only in the direction of $\overline{O'M}$, at the angle θ with $\cos\theta = \frac{O'M}{OM} = \frac{3}{5}$. Undoing the projection therefore multiplies areas by $\frac{5}{3}$, so the unpainted face has area $\frac{5}{3}\left(12\pi + 18\sqrt{3}\right) = 20\pi + 30\sqrt{3}$. Thus $a + b + c = 20 + 30 + 3 = 53$.

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