2014 AIME II Solutions

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1. Abe can paint the room in 15 hours, Bea can paint 50 percent faster than Abe, and Coe can paint twice as fast as Abe. Abe begins to paint the room and works alone for the first hour and a half. Then Bea joins Abe, and they work together until half the room is painted. Then Coe joins Abe and Bea, and they work together until the entire room is painted. Find the number of minutes after Abe begins for the three of them to finish painting the room.

Solution:

Abe paints $\frac{1}{900}$ of the room per minute, so Bea paints $\frac{3}{2} \cdot \frac{1}{900} = \frac{1}{600}$ per minute and Coe paints $\frac{2}{900} = \frac{1}{450}$ per minute. In the first 90 minutes Abe paints $\frac{90}{900} = \frac{1}{10}$ of the room.

Abe and Bea together paint $\frac{1}{900} + \frac{1}{600} = \frac{1}{360}$ per minute, and they must bring the total from $\frac{1}{10}$ up to $\frac{1}{2}$, which takes $\frac{2}{5} \cdot 360 = 144$ minutes. All three together paint $\frac{1}{360} + \frac{1}{450} = \frac{1}{200}$ per minute, so the remaining half of the room takes $\frac{1}{2} \cdot 200 = 100$ minutes.

The total time is $90 + 144 + 100 = 334$ minutes.

2. Arnold is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of men. For each of the three factors, the probability that a randomly selected man in the population has only this risk factor (and none of the others) is 0.1. For any two of the three factors, the probability that a randomly selected man has exactly these two risk factors (but not the third) is 0.14. The probability that a randomly selected man has all three risk factors, given that he has A and B, is $\frac{1}{3}$. The probability that a man has none of the three risk factors given that he does not have risk factor A is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

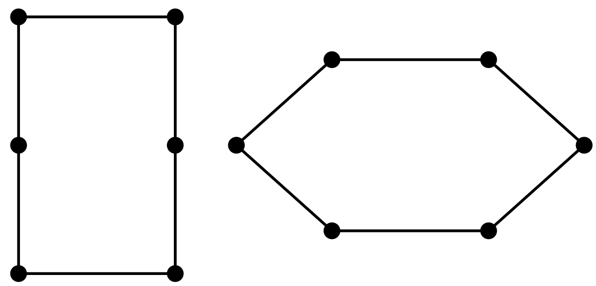
Solution:

Take a population of 100 men and fill in a Venn diagram. Each of the three exactly-one regions contains 10 men, and each of the three exactly-two regions contains 14. If x men have all three factors, then the men with both A and B number $x+14$, so the given conditional probability says $\frac{x}{x+14}=\frac{1}{3}$, giving $x=7$.

The union of the three sets therefore contains $3 \cdot 10+3 \cdot 14+7=79$ men, leaving 21 with no risk factor. The men with risk factor A number $10+14+14+7=45$, so 55 men do not have A.

The desired probability is $\frac{21}{55}$, which is in lowest terms, so $p+q=21+55=76$.

3. A rectangle has sides of length a and $36.$ A hinge is installed at each vertex of the rectangle and at the midpoint of each side of length $36.$ The sides of length a can be pressed toward each other keeping those two sides parallel so the rectangle becomes a convex hexagon as shown. When the figure is a hexagon with the sides of length a parallel and separated by a distance of $24,$ the hexagon has the same area as the original rectangle. Find $a^2.$



Solution:

In the hexagon, each side of length 36 has folded at its midpoint into two bars of length $18.$ The two sides of length a are 24 apart, so each bar spans a vertical distance of 12 and hence a horizontal distance of $\sqrt{18^2 - 12^2} = \sqrt{180} = 6\sqrt{5}.$

The line through the two midpoint hinges splits the hexagon into two congruent trapezoids with parallel sides a and $a + 12\sqrt{5}$ and height $12,$ so the hexagon has area

$$2 \cdot \frac{a + (a + 12\sqrt{5})}{2} \cdot 12 = 24a + 144\sqrt{5}.$$

Setting this equal to the rectangle's area $36a$ gives $12a = 144\sqrt{5},$ so $a = 12\sqrt{5}$ and $a^2 = 720.$

4. The repeating decimals $0.abab\overline{ab}$ and $0.abcabc\overline{abc}$ satisfy

$$0.abab\overline{ab} + 0.abcabc\overline{abc} = \frac{33}{37},$$

where $a, b,$ and c are (not necessarily distinct) digits. Find the three-digit number abc.

Solution:

Writing ab and abc for the two- and three-digit numbers, the decimals equal $\frac{ab}{99}$ and $\frac{abc}{999}$. Since $99 = 9 \cdot 11$ and $999 = 27 \cdot 37$, the common denominator is $27 \cdot 37 \cdot 11 = 10989$, and multiplying the equation by it gives

$$111 \cdot ab + 11 \cdot abc = \frac{33}{37} \cdot 10989 = 9801.$$

Modulo 11, since $9801 = 11 \cdot 891$ and $111 \equiv 1$, this forces $11 \mid ab$, so $a = b$. Then $ab = 11a$, and dividing the equation by 11 gives $111a + abc = 891$. Since $abc = 110a + c$, this is $221a + c = 891$, which requires $a = 4$ and $c = 7$.

Thus $a = b = 4, c = 7,$ and the three-digit number abc is 447.

5. Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

Solution:

Both cubics have zero x^2 coefficient, so their roots sum to 0 : the third root of p is $t = -r - s$, and the third root of q is $-(r + 4) - (s - 3) = t - 1$. The coefficient of x is a in both, so

$$rs + st + tr = (r + 4)(s - 3) + (s - 3)(t - 1) + (t - 1)(r + 4),$$

which simplifies to $t = 4r - 3s + 13$.

The constant terms give $b = -rst$ and $b + 240 = -(r + 4)(s - 3)(t - 1)$, so $240 = rst - (r + 4)(s - 3)(t - 1)$, i.e. $rs - 4st + 3tr - 3r + 4s + 12t - 252 = 0$. Substituting $t = 4r - 3s + 13$ reduces this to $12\left[(r - s)^2 + 7(r - s) - 8\right] = 0$, so $r - s = 1$ or $r - s = -8$.

If $r - s = 1$, then $t = 4r - 3s + 13 = r + 16$ and $t = -r - s = -2r + 1$, so $r = -5$: the roots are $-5, -6, 11$, and $b = -rst = -330$. If $r - s = -8$, then $t = r - 11 = -2r - 8$, so $r = 1$: the roots are $1, 9, -10$, and $b = 90$. The requested sum is $330 + 90 = 420$.

6. Charles has two six-sided dice. One of the dice is fair, and the other die is biased so that it comes up six with probability $\frac{2}{3}$, and each of the other five sides has probability $\frac{1}{15}$. Charles chooses one of the two dice at random and rolls it three times. Given that the first two rolls are both sixes, the probability that the third roll will also be a six is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

The desired conditional probability is

$$\frac{\Pr(\text{three sixes})}{\Pr(\text{first two are sixes})} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{6}\right)^3}{\frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{6}\right)^2},$$

since each die is chosen with probability $\frac{1}{2}$ and the fair die shows a six with probability $\frac{1}{6}$.

The numerator is $\frac{1}{2}\left(\frac{8}{27} + \frac{1}{216}\right) = \frac{65}{432}$ and the denominator is $\frac{1}{2}\left(\frac{4}{9} + \frac{1}{36}\right) = \frac{17}{72}$, so the probability is $\frac{65}{432} \cdot \frac{72}{17} = \frac{65}{102}$.

Since $65 = 5 \cdot 13$ and $102 = 2 \cdot 3 \cdot 17$ share no factor, $p+q = 65+102 = 167$.

7. Let $f(x) = \left(x^2+3x+2\right)^{\cos(\pi x)}$. Find the sum of all positive integers n for which

$$\left|\sum_{k=1}^{n} \log_{10} f(k)\right| = 1.$$

Solution:

Since $\cos(\pi k) = (-1)^k$ and $k^2+3k+2 = (k+1)(k+2)$, we have $f(k) = [(k+1)(k+2)]^{(-1)^k}$. The sum of the logarithms is the log of the product $\prod_{k=1}^{n} f(k)$, which telescopes: consecutive factors $\frac{1}{(k+1)(k+2)}$ and $(k+2)(k+3)$ leave only boundary terms.

For even n the product is $\frac{3 \cdot 4 \cdots (n+2)}{2 \cdot 3 \cdots (n+1)} = \frac{n+2}{2}$, and for odd n it is $\frac{3 \cdot 4 \cdots (n+1)}{2 \cdot 3 \cdots (n+2)} = \frac{1}{2(n+2)}$. The absolute value of the log equals 1 exactly when the product is 10 or $\frac{1}{10}$.

For even n, $\frac{n+2}{2} = 10$ gives $n = 18$; for odd n, $2(n+2) = 10$ gives $n=3$. The requested sum is $18+3 = 21$.

8. Circle C with radius 2 has diameter $\overline{AB}$. Circle D is internally tangent to circle C at A. Circle E is internally tangent to circle C, externally tangent to circle D, and tangent to $\overline{AB}$. The radius of circle D is three times the radius of circle E and can be written in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$.

Solution:

Let C, D, E also name the circles' centers, let s be the radius of circle E, so circle D has radius $3s$, and let F be the foot of E on $\overline{AB}$. Tangency gives $CE = 2 - s$, $DE = 3s + s = 4s$, and $EF = s$, while D lies on $\overline{AB}$ with $DC = 2 - 3s$.

Right triangles CEF and DEF give $CF = \sqrt{(2-s)^2 - s^2} = \sqrt{4 - 4s}$ and $DF = \sqrt{(4s)^2 - s^2} = s\sqrt{15}$. Since F is on the opposite side of C from A, we have $DF = DC + CF$, so

$$s\sqrt{15} = (2 - 3s) + \sqrt{4 - 4s}.$$

Moving $2 - 3s$ to the left and squaring gives $24s^2 - 8s = 2\sqrt{15}\, s\,(2 - 3s)$, i.e. $12s - 4 = \sqrt{15}\,(2 - 3s)$; squaring again yields $9s^2 + 84s - 44 = 0$, so $s = \frac{-14 + 4\sqrt{15}}{3}$. The radius of circle D is $3s = 4\sqrt{15} - 14 = \sqrt{240} - 14$, and $m + n = 240 + 14 = 254$.

9. Ten chairs are arranged in a circle. Find the number of subsets of this set of chairs that contain at least three adjacent chairs.

Solution:

The full set of 10 chairs qualifies; count the others by locating each maximal run of at least three adjacent chosen chairs at its clockwise start. Any such subset contains a block of four consecutive chairs that is empty-chosen-chosen-chosen. There are 10 positions for this block, and the remaining 6 chairs are free, giving $10 \cdot 2^6 = 640$.

This counts once for each maximal run of length at least 3. Two such runs require at least $3 + 3$ chosen chairs plus two gaps, so three runs are impossible, and subsets with exactly two runs are counted twice. To have two runs, place two disjoint empty-chosen-chosen-chosen blocks: $\frac{10 \cdot 3}{2} = 15$ ways (the second block fits in 3 positions among the remaining 6 chairs), with the last 2 chairs free, for $15 \cdot 2^2 = 60$ subsets.

The total is $1 + 640 - 60 = 581$.

10. Let z be a complex number with $|z| = 2014$. Let P be the polygon in the complex plane whose vertices are z and every w such that $\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}$. Then the area enclosed by P can be written in the form $n\sqrt{3}$, where n is an integer. Find the remainder when n is divided by 1000.

Solution:

Multiplying $\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}$ by $zw(z+w)$ gives $zw = (z+w)^2$, i.e. $z^2 + zw + w^2 = 0$. Multiplying by $z - w$ yields $z^3 - w^3 = 0$, so $w = \omega z$ or $w = \omega^2 z$, where ω is a primitive cube root of unity (and both indeed satisfy the original equation).

Thus P is the equilateral triangle with vertices $z, \omega z, \omega^2 z$, inscribed in the circle of radius 2014. Its area is

$$\frac{3\sqrt{3}}{4}(2014)^2 = 3 \cdot 1007^2\sqrt{3},$$

so $n = 3 \cdot 1007^2 = 3042147$.

The remainder when n is divided by 1000 is 147.

11. In $\triangle RED, RD = 1, \angle DRE = 75^\circ$ and $\angle RED = 45^\circ$. Let M be the midpoint of segment $\overline{RD}$. Point C lies on side $\overline{ED}$ such that $\overline{RC} \perp \overline{EM}$. Extend segment $\overline{DE}$ through E to point A such that $CA = AR$. Then $AE = \frac{a-\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer. Find $a+b+c$.

Solution:

Since $\angle RDE = 180^\circ - 75^\circ - 45^\circ = 60^\circ$, place $D = (0,0)$ with E on the positive x-axis, so $R = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The law of sines gives $DE = \frac{\sin 75^\circ}{\sin 45^\circ} = \frac{\sqrt{3}+1}{2}$, and $M = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$.

The slope of EM is $\frac{\sqrt{3}/4}{\frac{1}{4} - \frac{\sqrt{3}+1}{2}} = \frac{-\sqrt{3}}{1+2\sqrt{3}}$, so line RC has slope $\frac{1+2\sqrt{3}}{\sqrt{3}}$. Descending from R by $\frac{\sqrt{3}}{2}$ to the x-axis moves us left by $\frac{3/2}{1+2\sqrt{3}} = \frac{6\sqrt{3}-3}{22}$, so $C = (c, 0)$ with $c = \frac{1}{2} - \frac{6\sqrt{3}-3}{22} = \frac{7-3\sqrt{3}}{11}$.

For $A = (t, 0)$, the condition $CA = AR$ reads $(t-c)^2 = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4}$, which is linear in $t : t = \frac{1-c^2}{1-2c} = \frac{9+4\sqrt{3}}{11}$. Then

$$AE = t - \frac{\sqrt{3}+1}{2} = \frac{18 + 8\sqrt{3} - 11\sqrt{3} - 11}{22} = \frac{7 - 3\sqrt{3}}{22} = \frac{7 - \sqrt{27}}{22},$$

so $a + b + c = 7 + 27 + 22 = 56$.

12. Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A)+\cos(3B)+\cos(3C)=1.$ Two sides of the triangle have lengths 10 and $13.$ There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}.$ Find $m.$

Solution:

Using $1-\cos 3A = 2\sin^2\frac{3A}{2}$ and $\cos 3B+\cos 3C = 2\cos\frac{3(B+C)}{2}\cos\frac{3(B-C)}{2}$, together with $\frac{3(B+C)}{2} = 270^\circ - \frac{3A}{2}$ so that $\cos\frac{3(B+C)}{2} = -\sin\frac{3A}{2}$, the condition becomes

$$0 = 2\sin\frac{3A}{2}\left(\sin\frac{3A}{2}+\cos\frac{3(B-C)}{2}\right) = 2\sin\frac{3A}{2}\left(\cos\frac{3(B-C)}{2}-\cos\frac{3(B+C)}{2}\right) = 4\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}.$$

For an angle X of a triangle, $\frac{3X}{2}$ lies strictly between 0° and 270°, so $\sin\frac{3X}{2}=0$ exactly when $X = 120^\circ$. Hence one angle of the triangle is 120°.

The remaining side is longest when the 120° angle sits between the sides of lengths 10 and 13 (if 120° were opposite one of them, the remaining side would be shorter than that side). By the law of cosines its length is $\sqrt{10^2+13^2+10\cdot 13} = \sqrt{399}$, so $m = 399.$

13. Ten adults enter a room, remove their shoes, and toss their shoes into a pile. Later, a child randomly pairs each left shoe with a right shoe without regard to which shoes belong together. The probability that for every positive integer $k<5,$ no collection of k pairs made by the child contains the shoes from exactly k of the adults is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n.$

Solution:

The child's pairing matches left shoe j with right shoe $\pi(j)$ for a uniformly random permutation π of $\{1,\ldots,10\}$. A collection of k pairs uses k left and k right shoes, so it involves exactly k adults precisely when those adults' indices are closed under π — that is, when the collection is a union of cycles of π. The condition therefore says π has no cycle of length less than 5.

The cycle lengths must partition 10 into parts of size at least 5 : either one 10-cycle or two 5-cycles. There are $9!$ ten-cycles, and $\frac{1}{2}\binom{10}{5}(4!)^2 = \frac{9!}{5}$ permutations that are products of two 5-cycles.

The probability is

$$\frac{9!+\frac{1}{5}\cdot 9!}{10!} = \frac{1+\frac{1}{5}}{10} = \frac{3}{25},$$

so $m+n = 3+25 = 28.$

14. In $\triangle ABC$, $AB = 10$, $\angle A = 30^\circ$, and $\angle C = 45^\circ$. Let H, D, and M be points on line $\overline{BC}$ such that $\overline{AH} \perp \overline{BC}$, $\angle BAD = \angle CAD$, and $BM = CM$. Point N is the midpoint of segment $\overline{HM}$, and point P is on ray AD such that $\overline{PN} \perp \overline{BC}$. Then $AP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Let ray AD meet the circumcircle of $\triangle ABC$ again at E. Since AD bisects angle A, the point E is the midpoint of arc BC, so E lies on the perpendicular bisector of $\overline{BC}$ and projects onto line BC at M. The projections of the collinear points A, P, E onto line BC are H, N, M, and projection preserves ratios along a line; since N is the midpoint of $\overline{HM}$, point P is the midpoint of $\overline{AE}$.

Here $\angle B = 105^\circ$, and $\angle CBE = \angle CAE = 15^\circ$ (both subtend arc CE), so $\angle ABE = 120^\circ$. Also $\angle AEB = \angle ACB = 45^\circ$ (both subtend arc AB). The law of sines in $\triangle ABE$ gives

$$AE = AB \cdot \frac{\sin \angle ABE}{\sin \angle AEB} = 10 \cdot \frac{\sin 120^\circ}{\sin 45^\circ} = 5\sqrt{6}.$$

Therefore $AP = \frac{1}{2}AE = \frac{5\sqrt{6}}{2}$, so $AP^2 = \frac{75}{2}$ and $m + n = 75 + 2 = 77$.

15. For any integer $k \geq 1$, let $p(k)$ be the smallest prime which does not divide k. Define the integer function $X(k)$ to be the product of all primes less than $p(k)$ if $p(k) > 2$, and $X(k) = 1$ if $p(k) = 2$. Let $\{x_n\}$ be the sequence defined by $x_0 = 1$, and $x_{n+1}X(x_n) = x_n p(x_n)$ for $n \geq 0$. Find the smallest positive integer t such that $x_t = 2090$.

Solution:

List the primes in order as $\rho_0 = 2, \rho_1 = 3, \rho_2 = 5, \ldots$. Every x_n is squarefree, so it is described by the set of primes dividing it, and we claim this set encodes n in binary: if $n = \sum_i d_i 2^i$ with $d_i \in \{0, 1\}$, then $x_n = \prod_i \rho_i^{d_i}$.

Indeed, suppose $x_n = \prod_i \rho_i^{d_i}$ and let j be the smallest index with $d_j = 0$. Then $p(x_n) = \rho_j$, and $X(x_n) = \rho_0 \rho_1 \cdots \rho_{j-1}$ is exactly the product of the primes for the trailing 1-bits (with $X(x_n) = 1$ when $j = 0$). So

$$x_{n+1} = \frac{x_n \, \rho_j}{\rho_0 \rho_1 \cdots \rho_{j-1}}$$

removes the trailing ones and inserts ρ_j — precisely adding 1 in binary. Since $x_0 = 1$ corresponds to 0, induction proves the claim.

Now $2090 = 2 \cdot 5 \cdot 11 \cdot 19 = \rho_0 \rho_2 \rho_4 \rho_7$, which corresponds to binary digits at positions $0, 2, 4, 7$. Hence $t = 2^0 + 2^2 + 2^4 + 2^7 = 149$.

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