2014 AIME I Solutions

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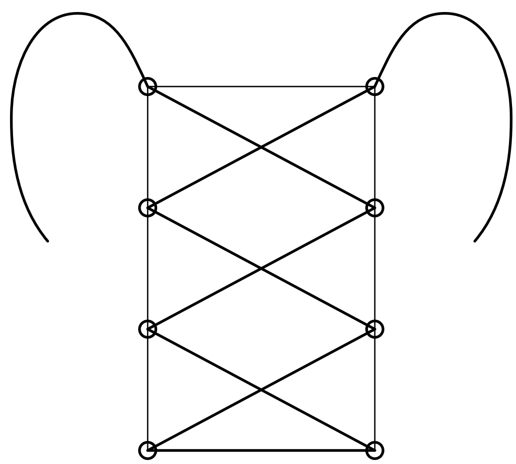
1. The 8 eyelets for the lace of a sneaker all lie on a rectangle, four equally spaced on each of the longer sides. The rectangle has a width of 50 mm and a length of 80 mm. There is one eyelet at each vertex of the rectangle. The lace itself must pass between the vertex eyelets along a width side of the rectangle and then crisscross between successive eyelets until it reaches the two eyelets at the other width side of the rectangle as shown. After passing through these final eyelets, each of the ends of the lace must extend at least 200 mm farther to allow a knot to be tied. Find the minimum length of the lace in millimeters.



Solution:

The four eyelets on each 80 mm side are equally spaced with one at each vertex, so consecutive eyelets on a side are $\frac{80}{3}$ mm apart. The lace consists of one segment across the 50 mm width, six crisscross pieces (after the width crossing, each of the two strands makes three crossings to reach the top), and two free ends of at least 200 mm each. The lace is shortest when every piece is a straight segment.

Each crisscross piece spans the full width and rises one gap, so its length is

$$\sqrt{50^2 + \left(\tfrac{80}{3}\right)^2} = \sqrt{\tfrac{28900}{9}} = \frac{170}{3}.$$

The minimum length is $50 + 6 \cdot \frac{170}{3} + 2 \cdot 200 = 50 + 340 + 400 = 790.$

2. An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

Solution:

Both balls are green with probability $\frac{4}{10} \cdot \frac{16}{16+N}$, and both are blue with probability $\frac{6}{10} \cdot \frac{N}{16+N}$. The condition is

$$\frac{64+6N}{10(16+N)} = \frac{29}{50}.$$

Clearing denominators, $5(64+6N) = 29(16+N)$, so $320 + 30N = 464 + 29N$, giving $N = 144$.

3. Find the number of rational numbers r, $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and the denominator have a sum of 1000.

Solution:

Write $r = \frac{a}{b}$ in lowest terms with $a + b = 1000$; since $0 < r < 1$, we need $1 \leq a \leq 499$. Because $\gcd(a,b) = \gcd(a, 1000-a) = \gcd(a, 1000)$, the fraction is in lowest terms exactly when a is coprime to 1000.

There are $\varphi(1000) = 1000 \cdot \frac{1}{2} \cdot \frac{4}{5} = 400$ integers in $[1, 999]$ coprime to 1000, and they pair up as $a \leftrightarrow 1000 - a$ (note $a = 500$ is not coprime to 1000), so exactly 200 of them are less than 500. The answer is 200.

4. Jon and Steve ride their bicycles on a path that parallels two side-by-side train tracks running in the east/west direction. Jon rides east at 20 miles per hour, and Steve rides west at 20 miles per hour. Two trains of equal length, traveling in opposite directions at constant but different speeds, each pass the two riders. Each train takes exactly 1 minute to go past Jon. The westbound train takes 10 times as long as the eastbound train to go past Steve. The length of each train is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Let the eastbound and westbound trains have speeds v_1 and v_2 miles per hour and common length L miles. A train passes a rider in time L divided by their relative speed. Passing Jon (riding east at 20) in $\frac{1}{60}$ hour gives

$$\frac{L}{v_1 - 20} = \frac{L}{v_2 + 20} = \frac{1}{60},$$

so $v_1 = 60L + 20$ and $v_2 = 60L - 20$.

Relative to Steve (riding west at 20), the speeds are $v_1 + 20$ and $v_2 - 20$, and the westbound train takes 10 times as long: $\frac{L}{v_2 - 20} = \frac{10L}{v_1 + 20}$, so $v_1 + 20 = 10(v_2 - 20)$. Substituting, $60L + 40 = 600L - 400$, so $540L = 440$ and $L = \frac{22}{27}$.

Since $\gcd(22, 27) = 1$, the answer is $22 + 27 = 49$.

5. Let the set $S = \{P_1, P_2, \ldots, P_{12}\}$ consist of the twelve vertices of a regular 12-gon. A subset Q of S is called *communal* if there is a circle such that all points of Q are inside the circle, and all points of S not in Q are outside of the circle. How many communal subsets are there? (Note that the empty set is a communal subset.)

Solution:

A subset Q is communal exactly when its vertices are consecutive around the 12-gon. Indeed, a separating circle meets the circumcircle of the 12-gon in at most two points, so the vertices inside it form a contiguous arc. Conversely, any run of consecutive vertices can be separated from the remaining vertices by a line, and a sufficiently large circle on the proper side of that line contains exactly that run.

For each size k with $1 \leq k \leq 11$ there are 12 runs of k consecutive vertices (one starting at each vertex), giving $12 \cdot 11 = 132$ subsets, and the empty set and all of S are also communal. The total is $132 + 2 = 134.$

6. The graphs $y = 3(x-h)^2 + j$ and $y = 2(x-h)^2 + k$ have y-intercepts of 2013 and $2014,$ respectively, and each graph has two positive integer x-intercepts. Find $h.$

Solution:

Setting $x = 0$ gives $3h^2 + j = 2013$ and $2h^2 + k = 2014.$ Expanding, the first graph is $y = 3x^2 - 6hx + 2013,$ whose roots are positive integers with sum $2h$ and product $\frac{2013}{3} = 671 = 11 \cdot 61.$ Similarly the second is $y = 2x^2 - 4hx + 2014,$ with integer roots of sum $2h$ and product $\frac{2014}{2} = 1007 = 19 \cdot 53.$

The first pair of roots is $\{11, 61\}$ or $\{1, 671\},$ so $2h = 72$ or $672;$ the second pair is $\{19, 53\}$ or $\{1, 1007\},$ so $2h = 72$ or $1008.$ The only common value is $2h = 72,$ so $h = 36,$ which indeed gives x-intercepts $11, 61$ and $19, 53.$

7. Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = \arg\left(\frac{w-z}{z}\right)$. The maximum possible value of $\tan^2\theta$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$. (Note that $\arg(w)$, for $w \neq 0$, denotes the measure of the angle that the ray from 0 to w makes with the positive real axis in the complex plane.)

Solution:

Since $\frac{w-z}{z} = \frac{w}{z} - 1$, and $\frac{w}{z}$ can be any complex number of modulus $\frac{1}{10}$, the point $\zeta = \frac{w-z}{z}$ ranges over the circle of radius $\frac{1}{10}$ centered at -1.

Because $\tan^2\theta$ is unchanged when θ shifts by 180°, we want the largest angle α that a ray from the origin to this circle makes with the real axis. The extreme rays are tangent to the circle, where $\sin\alpha = \frac{1/10}{1} = \frac{1}{10}$.

Then

$$\tan^2\theta = \frac{\sin^2\alpha}{1-\sin^2\alpha} = \frac{1/100}{99/100} = \frac{1}{99},$$

so $p+q = 1+99 = 100$.

8. The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

Solution:

The condition is $N^2 \equiv N \pmod{10^4}$, that is, $N(N-1) \equiv 0 \pmod{2^4 \cdot 5^4}$. Since consecutive integers are coprime, 16 divides one of $N, N-1$ and 625 divides one of them. This gives four cases modulo $10000 : N \equiv 0, N \equiv 1, N \equiv 625$ (which is 0 mod 625 and 1 mod 16), and $N \equiv 9376$ (which is 0 mod 16 and 1 mod 625).

The last four digits $abcd$ must have $a \neq 0$, which rules out $0000, 0001,$ and $0625.$ So $abcd = 9376$ – for instance $9376^2 = 87909376$ – and $abc = 937.$

9. Let $x_1 < x_2 < x_3$ be the three real roots of the equation $\sqrt{2014}\, x^3 - 4029x^2 + 2 = 0$. Find $x_2(x_1 + x_3)$.

Solution:

Write $a = \sqrt{2014}$, so the equation is $ax^3 - (2a^2 + 1)x^2 + 2 = 0$. It factors as

$$(ax - 1)\left(x^2 - 2ax - 2\right) = 0,$$

as expanding confirms. So one root is $\frac{1}{a}$, and the other two are $a \pm \sqrt{a^2 + 2}$, with product -2 and sum $2a$.

Since $a - \sqrt{a^2 + 2} < 0 < \frac{1}{a} < a + \sqrt{a^2 + 2}$, the middle root is $x_2 = \frac{1}{a}$, and $x_1 + x_3 = 2a$. Therefore $x_2(x_1 + x_3) = \frac{1}{a} \cdot 2a = 2.$

10. A disk with radius 1 is externally tangent to a disk with radius 5. Let A be the point where the disks are tangent, C be the center of the smaller disk, and E be the center of the larger disk. While the larger disk remains fixed, the smaller disk is allowed to roll along the outside of the larger disk until the smaller disk has turned through an angle of 360°. That is, if the center of the smaller disk has moved to the point D, and the point on the smaller disk that began at A has now moved to point B, then $\overline{AC}$ is parallel to $\overline{BD}$. Then $\sin^2(\angle BEA) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Place E at the origin with $C = (6, 0)$, so $A = (5, 0)$. When a circle of radius 1 rolls without slipping outside a fixed circle of radius 5 and its center sweeps an angle φ about E, the rolling contact turns the disk through 5φ relative to the line of centers, and the revolution of that line adds φ more, so the disk turns 6φ in the ground frame. Turning through 360° therefore means $\varphi = 60^\circ$, so $D = 6(\cos 60^\circ, \sin 60^\circ) = (3, 3\sqrt{3})$.

Having turned through a full 360°, the disk is back in its original orientation, so the vector from its center to the marked point is unchanged: $B = D + (A - C) = (3 - 1, 3\sqrt{3}) = (2, 3\sqrt{3})$. (In particular $\overline{BD}$ is parallel to $\overline{AC}$, as the problem states.)

The ray EA is the positive x-axis, so

$$\sin^2(\angle BEA) = \frac{(3\sqrt{3})^2}{2^2 + (3\sqrt{3})^2} = \frac{27}{31},$$

and $m + n = 27 + 31 = 58$.

11. A token starts at the point $(0,0)$ of an xy-coordinate grid and then makes a sequence of six moves. Each move is 1 unit in a direction parallel to one of the coordinate axes. Each move is selected randomly from the four possible directions and independently of the other moves. The probability that the token ends at a point on the graph of $|y|=|x|$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Work in the diagonal coordinates $u=x+y$ and $v=x-y$. Each of the four moves changes u by ±1 and v by ±1, and the four moves realize all four sign combinations equally often — so u and v perform independent six-step ±1 walks. The token ends on $|y|=|x|$ exactly when $y=\pm x$, that is, when $u=0$ or $v=0$.

Each of $u=0$ and $v=0$ requires three $+1$s and three -1s, with probability $\binom{6}{3}/2^6=\frac{20}{64}=\frac{5}{16}$. By independence and inclusion-exclusion, the probability is

$$\frac{5}{16}+\frac{5}{16}-\left(\frac{5}{16}\right)^2=\frac{160-25}{256}=\frac{135}{256}.$$

Thus $m+n=135+256=391.$

12. Let $A = \{1, 2, 3, 4\}$, and let f and g be randomly chosen (not necessarily distinct) functions from A to A. The probability that the range of f and the range of g are disjoint is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Solution:

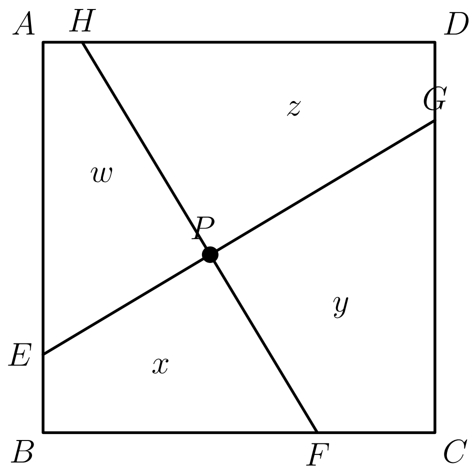
Condition on the range of f. If it has k elements, then the range of g is disjoint from it exactly when g maps A into the remaining $4 - k$ elements, which happens for $(4 - k)^4$ of the $4^4 = 256$ functions g.

Count functions f by range size: 4 constant functions; $\binom{4}{2}(2^4 - 2) = 84$ with range size 2; $\binom{4}{3} \cdot 36 = 144$ with range size 3 (there are 36 surjections from four elements onto three); and $4! = 24$ bijections. The number of favorable pairs is

$$4 \cdot 3^4 + 84 \cdot 2^4 + 144 \cdot 1^4 + 24 \cdot 0^4 = 324 + 1344 + 144 = 1812.$$

The probability is $\frac{1812}{4^8} = \frac{1812}{65536} = \frac{453}{16384}$, and since 16384 is a power of 2 while $453 = 3 \cdot 151$ is odd, this is in lowest terms. Thus $m = 453$.

13. On square $ABCD$, points E, F, G, and H lie on sides $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$, respectively, so that $\overline{EG} \perp \overline{FH}$ and $EG = FH = 34$. Segments $\overline{EG}$ and $\overline{FH}$ intersect at a point P, and the areas of the quadrilaterals $AEPH, BFPE, CGPF$, and $DHPG$ are in the ratio $269 : 275 : 405 : 411$. Find the area of square $ABCD$.



$w : x : y : z = 269 : 275 : 405 : 411$

Solution:

Place $B = (0,0), C = (s,0), D = (s,s), A = (0,s)$, with $E = (0,e), F = (f,0)$, $G = (s,g), H = (h,s)$. The regions $AEPH$ and $DHPG$ together form the trapezoid $AEGD$, of area $\frac{s\,((s-e)+(s-g))}{2}$; since $\frac{269+411}{1360} = \frac{1}{2}$, this is half of s^2, forcing $e + g = s$, i.e. EG passes through the center. Likewise $AEPH$ and $BFPE$ form the trapezoid $ABFH$ of area $\frac{s(f+h)}{2} = \frac{269+275}{1360}\,s^2 = \frac{2}{5}s^2$, so $f + h = \frac{4s}{5}$.
Perpendicularity of the directions $(s, g - e)$ and $(h - f, s)$ gives $h - f = e - g$.
Writing $\delta = g - e$, we get $E = \left(0, \frac{s-\delta}{2}\right), G = \left(s, \frac{s+\delta}{2}\right), F = \left(\frac{2s}{5} + \frac{\delta}{2}, 0\right), H = \left(\frac{2s}{5} - \frac{\delta}{2}, s\right)$, and $EG = 34$ gives $s^2 + \delta^2 = 1156$.

Intersecting lines EG and FH (and simplifying with $s^2 + \delta^2 = 1156$) yields

$$P = \left(\frac{s}{2} - \frac{s^3}{11560}, \frac{s}{2} - \frac{s^2\delta}{11560}\right),$$

and the shoelace formula on A, E, P, H then gives $[AEPH] = \frac{s^2}{5} - \frac{s^3\delta}{231200}$. Setting this equal to $\frac{269}{1360}s^2$ leaves $\frac{3s^2}{1360} = \frac{s^3\delta}{231200}$, so $s\delta = 510$.

Now $s^2 + \delta^2 = 1156$ and $s^2\delta^2 = 260100$, so s^2 and δ^2 are the roots of $t^2 - 1156t + 260100 = 0$, which are $\frac{1156 \pm 544}{2} = 850$ and 306. Since $|\delta| = |g - e| < s$, the area is $s^2 = 850$.

14. Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4.$$

There are positive integers a, b, and c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

Solution:

Add 4 to both sides, giving one unit to each fraction: since $\frac{k}{x-k} + 1 = \frac{x}{x-k}$, the equation becomes

$$x\left(\frac{1}{x-3} + \frac{1}{x-5} + \frac{1}{x-17} + \frac{1}{x-19}\right) = x^2 - 11x = x(x-11).$$

Besides $x = 0$, we can divide by x and substitute $t = x - 11$, which pairs the fractions symmetrically:

$$\frac{2t}{t^2 - 64} + \frac{2t}{t^2 - 36} = t.$$

Besides $t = 0$, dividing by t gives $\frac{2}{t^2-64} + \frac{2}{t^2-36} = 1$. With $u = t^2$, clearing denominators gives $2(u - 36) + 2(u - 64) = (u - 36)(u - 64)$, i.e. $u^2 - 104u + 2504 = 0$, so $u = 52 \pm \sqrt{200}$.

The largest solution is $m = 11 + \sqrt{52 + \sqrt{200}} \approx 19.1$, which exceeds the other candidates $0, 11$, and $11 \pm \sqrt{52 \pm \sqrt{200}}$. Therefore $a + b + c = 11 + 52 + 200 = 263$.

15. In $\triangle ABC$, $AB = 3$, $BC = 4$, and $CA = 5$. Circle ω intersects $\overline{AB}$ at E and B, $\overline{BC}$ at B and D, and $\overline{AC}$ at F and G. Given that $EF = DF$ and $\frac{DG}{EG} = \frac{3}{4}$, length $DE = \frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. Find $a + b + c$.

Solution:

Since $3^2 + 4^2 = 5^2$, angle B is right, and as $\angle EBD = 90^\circ$ is inscribed in ω, the chord ED is a diameter. Hence $\angle EFD = \angle EGD = 90^\circ$. From $EF = DF$, triangle EFD is an isosceles right triangle, so $EF = DF = \frac{DE}{\sqrt{2}}$ and $\angle FED = 45^\circ$; from $DG : EG = 3 : 4$ and $DG^2 + EG^2 = DE^2$ we get $DG = \frac{3}{5}DE$ and $EG = \frac{4}{5}DE$. On ω the order is E, F, G, D (both F and G lie on the arc opposite B, and $\angle FED = 45^\circ$ exceeds $\angle GED = \arcsin\frac{3}{5}$, so F is farther from D).

Line AC is the line FG, so the distances from E and D to it follow from the angles of cyclic quadrilateral $EFGD$. At F : $\angle EFG = 180^\circ - \angle GDE$ and $\sin\angle GDE = \frac{EG}{DE} = \frac{4}{5}$, so the distance from E is $EF \sin\angle EFG = \frac{DE}{\sqrt{2}} \cdot \frac{4}{5} = \frac{2\sqrt{2}}{5}DE$. At G : $\angle FGD = 180^\circ - \angle FED = 135^\circ$, so the distance from D is $DG \sin\angle FGD = \frac{3}{5}DE \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{10}DE.$

Now place $B = (0,0)$, $C = (4,0)$, $A = (0,3)$, so $E = (0,e)$, $D = (d,0)$, line AC : $3x + 4y = 12$, and $DE^2 = d^2 + e^2$. Setting $k = \frac{\sqrt{2}}{2}DE$, the two distance formulas read $\frac{12-4e}{5} = \frac{2\sqrt{2}}{5}DE$ and $\frac{12-3d}{5} = \frac{3\sqrt{2}}{10}DE$, which give $e = 3 - k$ and $d = 4 - k$. Then $DE^2 = 2k^2$ becomes $2k^2 = (3-k)^2 + (4-k)^2 = 2k^2 - 14k + 25$, so $k = \frac{25}{14}$ and $DE = \sqrt{2}\,k = \frac{25\sqrt{2}}{14}$. Therefore $a + b + c = 25 + 2 + 14 = 41.$

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