2013 AIME II Solutions

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1. Suppose that the measurement of time during the day is converted to the metric system so that each day has 10 metric hours, and each metric hour has 100 metric minutes. Digital clocks would then be produced that would read $9{:}99$ just before midnight, $0{:}00$ at midnight, $1{:}25$ at the former $3{:}00$ AM, and $7{:}50$ at the former $6{:}00$ PM. After the conversion, a person who wanted to wake up at the equivalent of the former $6{:}36$ AM would set his new digital alarm clock for $A{:}BC$, where A, B, and C are digits. Find $100A + 10B + C$.

Solution:

An ordinary day has $60 \cdot 24 = 1440$ minutes, and $6{:}36$ AM comes $6 \cdot 60 + 36 = 396$ minutes after midnight. A metric day has $10 \cdot 100 = 1000$ metric minutes, so the equivalent metric time is

$$\frac{396}{1440} \cdot 1000 = 275$$

metric minutes after midnight, which the new clock displays as $2{:}75$.

Therefore $100A + 10B + C = 275$.

2. Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a+b$.

Solution:

Working from the outside in, $\log_2(\cdot) = 0$ forces $\log_{2^a}(\log_{2^b}(2^{1000})) = 1$, so $\log_{2^b}(2^{1000}) = 2^a$, so $2^{1000} = (2^b)^{2^a} = 2^{b\cdot 2^a}$. Hence

$$b \cdot 2^a = 1000 = 2^3 \cdot 125.$$

Since $a \geq 1$ and b is a positive integer, 2^a must be one of $2, 4, 8,$ giving $(a,b) = (1,500)$, $(2,250), (3,125)$. The sum of all possible values of $a+b$ is $501 + 252 + 128 = 881.$

3. A large candle is 119 centimeters tall. It is designed to burn down more quickly when it is first lit and more slowly as it approaches its bottom. Specifically, the candle takes 10 seconds to burn down the first centimeter from the top, 20 seconds to burn down the second centimeter, and $10k$ seconds to burn down the k-th centimeter. Suppose it takes T seconds for the candle to burn down completely. Then $\frac{T}{2}$ seconds after it is lit, the candle's height in centimeters will be $h.$ Find $10h.$

Solution:

Burning the first x centimeters takes $10(1+2+\cdots+x) = 5x(x+1)$ seconds, so $T = 5 \cdot 119 \cdot 120 = 71400$ and $\frac{T}{2} = 35700.$

Setting $5x(x+1) = 35700$ gives $x(x+1) = 7140 = 84 \cdot 85,$ so at time $\frac{T}{2}$ the candle has burned down exactly 84 centimeters. Its height is $h = 119 - 84 = 35,$ and $10h = 350.$

4. In the Cartesian plane let $A = (1, 0)$ and $B = (2, 2\sqrt{3})$. Equilateral triangle ABC is constructed so that C lies in the first quadrant. Let $P = (x, y)$ be the center of $\triangle ABC$. Then $x \cdot y$ can be written as $\frac{p\sqrt{q}}{r}$, where p and r are relatively prime positive integers and q is an integer that is not divisible by the square of any prime. Find $p + q + r$.

Solution:

The midpoint of $\overline{AB}$ is $M = \left(\frac{3}{2}, \sqrt{3}\right)$, and $AB = \sqrt{1+12} = \sqrt{13}$. The third vertex lies at distance $\frac{\sqrt{3}}{2}\sqrt{13}$ from M along a direction perpendicular to $\overrightarrow{AB} = (1, 2\sqrt{3})$; a unit perpendicular is $\frac{1}{\sqrt{13}}(2\sqrt{3}, -1)$. Taking the sign that lands in the first quadrant,

$$C = M + \frac{\sqrt{3}}{2}(2\sqrt{3}, -1) = \left(\frac{9}{2}, \frac{\sqrt{3}}{2}\right)$$

(the other choice has negative x-coordinate).

The center of an equilateral triangle is its centroid, the average of the vertices:

$$P = \left(\frac{1 + 2 + \frac{9}{2}}{3}, \frac{0 + 2\sqrt{3} + \frac{\sqrt{3}}{2}}{3}\right) = \left(\frac{5}{2}, \frac{5\sqrt{3}}{6}\right).$$

Then $x \cdot y = \frac{5}{2} \cdot \frac{5\sqrt{3}}{6} = \frac{25\sqrt{3}}{12}$, so $p + q + r = 25 + 3 + 12 = 40$.

5. In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a+b+c$.

Solution:

Scale so the side length is 6, with $BD = DE = EC = 2$. In triangle AEC, the law of cosines gives

$$AE^2 = 6^2 + 2^2 - 2 \cdot 6 \cdot 2 \cos 60^\circ = 28,$$

so $AE = 2\sqrt{7}$, and $AD = 2\sqrt{7}$ by symmetry.

Since DE is one third of BC and triangles ADE and ABC share the apex A, we get $[ADE] = \frac{1}{3}[ABC] = \frac{1}{3} \cdot \frac{36\sqrt{3}}{4} = 3\sqrt{3}$. On the other hand $[ADE] = \frac{1}{2} \cdot AD \cdot AE \cdot \sin(\angle DAE) = 14\sin(\angle DAE)$.

Therefore $\sin(\angle DAE) = \frac{3\sqrt{3}}{14}$, and $a+b+c = 3+3+14 = 20$.

6. Find the least positive integer N such that the set of 1000 consecutive integers beginning with $1000 \cdot N$ contains no square of an integer.

Solution:

The block $\{1000N, \ldots, 1000N + 999\}$ misses all squares exactly when some consecutive squares x^2 and $(x+1)^2$ jump over it, which requires $(x+1)^2 - x^2 = 2x + 1 > 1000$, so $x \geq 500$. In particular every block below $500^2 = 250000$ contains a square, so we search from there.

Write $x = 500 + a$ with $a \geq 0$. Then $x^2 = 1000(250 + a) + a^2$, so as long as $a^2 < 1000$ (that is, $a \leq 31$), the square x^2 lies in block $250 + a$; these cover blocks 250 through 281. Block $251 + a$ is skipped exactly when

$$(x+1)^2 = 1000(250 + a) + a^2 + 2a + 1001 \geq 1000(252 + a),$$

that is, $a^2 + 2a \geq 999$. For $a \leq 30$ this fails (so $(x+1)^2$ lands in block $251 + a$), and it first holds at $a = 31$, since $961 + 62 = 1023$.

Indeed $531^2 = 281961$ and $532^2 = 283024$ straddle the block starting at 282000. The least such N is $251 + 31 = 282$.

7. A group of clerks is assigned the task of sorting 1775 files. Each clerk sorts at a constant rate of 30 files per hour. At the end of the first hour, some of the clerks are reassigned to another task; at the end of the second hour, the same number of the remaining clerks are also reassigned to another task, and a similar reassignment occurs at the end of the third hour. The group finishes the sorting in 3 hours and 10 minutes. Find the number of files sorted during the first one and a half hours of sorting.

Solution:

Let n clerks start and k be reassigned at the end of each hour. In the final 10 minutes each remaining clerk sorts 5 files, so

$$30n + 30(n-k) + 30(n-2k) + 5(n-3k) = 1775,$$

which simplifies to $95n - 105k = 1775$, or $19n - 21k = 355$.

Modulo 19 this reads $-2k \equiv 13$, so $2k \equiv 6$ and $k \equiv 3 \pmod{19}$. Taking $k = 3$ gives $n = \frac{355+63}{19} = 22$; the next candidate, $k = 22$, gives $n = 43$, for which $n - 3k < 0$. So $n = 22$ and $k = 3$.

In the first hour 22 clerks sort $30 \cdot 22 = 660$ files, and in the next half hour 19 clerks sort $15 \cdot 19 = 285$, for a total of $660 + 285 = 945$.

8. A hexagon that is inscribed in a circle has side lengths $22, 22, 20, 22, 22,$ and 20 in that order. The radius of the circle can be written as $p + \sqrt{q}$, where p and q are positive integers. Find $p + q$.

Solution:

Let r be the radius, and let each chord of length 22 subtend central angle α and each chord of length 20 subtend β. The six central angles fill the circle: $4\alpha + 2\beta = 360^\circ$, so $\frac{\beta}{2} = 90^\circ - \alpha$ and $\sin\frac{\beta}{2} = \cos\alpha$.

Half a 20-chord gives $\sin\frac{\beta}{2} = \frac{10}{r}$, and the law of cosines on the isosceles triangle with legs r and base 22 gives $22^2 = 2r^2(1 - \cos\alpha)$, so $\cos\alpha = 1 - \frac{242}{r^2}$. Equating,

$$1 - \frac{242}{r^2} = \frac{10}{r} \quad \Longrightarrow \quad r^2 - 10r - 242 = 0,$$

so $r = 5 + \sqrt{267}$ (taking the positive root).

Therefore $p + q = 5 + 267 = 272.$

9. A 7×1 board is completely covered by $m \times 1$ tiles without overlap; each tile may cover any number of consecutive squares, and each tile lies completely on the board. Each tile is either red, blue, or green. Let N be the number of tilings of the 7×1 board in which all three colors are used at least once. For example, a 1×1 red tile followed by a 2×1 green tile, a 1×1 green tile, a 2×1 blue tile, and a 1×1 green tile is a valid tiling. Note that if the 2×1 blue tile is replaced by two 1×1 blue tiles, this results in a different tiling. Find the remainder when N is divided by 1000.

Solution:

First count colored tilings when k colors are available. The first square's tile can be colored in k ways, and each of the remaining 6 squares either extends the current tile or starts a new tile in one of the k colors, giving $k+1$ choices per square. So there are $k(k+1)^6$ tilings.

With three colors that is $3 \cdot 4^6 = 12288$ tilings. By inclusion-exclusion over the unused colors, the number using all three colors is

$$N = 3 \cdot 4^6 - 3 \cdot (2 \cdot 3^6) + 3 \cdot (1 \cdot 2^6) = 12288 - 4374 + 192 = 8106.$$

The remainder when N is divided by 1000 is 106.

10. Given a circle of radius $\sqrt{13}$, let A be a point at a distance $4+\sqrt{13}$ from the center O of the circle. Let B be the point on the circle nearest to point A. A line passing through the point A intersects the circle at points K and L. The maximum possible area for $\triangle BKL$ can be written in the form $\frac{a-b\sqrt{c}}{d}$, where $a, b, c,$ and d are positive integers, a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

Solution:

The nearest point B lies on segment OA with $OB=\sqrt{13}$ and $AB=4$. Triangles OKL and BKL share the base KL, and their heights are the distances from O and B to the line through A. For any point P on line OA, that distance is $PA\sin\varphi$, where φ is the angle between the two lines, so

$$\frac{[BKL]}{[OKL]}=\frac{AB}{AO}=\frac{4}{4+\sqrt{13}}.$$

Since $OK=OL=\sqrt{13}$, we have $[OKL]=\frac{13}{2}\sin(\angle KOL)\leq\frac{13}{2}$, with equality when $\angle KOL=90^\circ$. Such a chord lies at distance $\sqrt{13/2}$ from O, which is less than OA, so a line through A can achieve it.

The maximum area is

$$[BKL]=\frac{13}{2}\cdot\frac{4}{4+\sqrt{13}}=\frac{26}{4+\sqrt{13}}=\frac{26(4-\sqrt{13})}{3}=\frac{104-26\sqrt{13}}{3},$$

so $a+b+c+d=104+26+13+3=146.$

11. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, and let N be the number of functions f from set A to set A such that $f(f(x))$ is a constant function. Find the remainder when N is divided by 1000.

Solution:

Say $f(f(x)) = a$ for all x, and let $S = \{x : f(x) = a\}$. Picking any $t \in S$, we get $a = f(f(t)) = f(a)$, so $a \in S$. Every x satisfies $f(x) \in S$ (because $f(f(x)) = a$), and if $x \notin S$ then $f(x) \neq a$, so f maps the complement of S into $S \setminus \{a\}$. Conversely, any f built this way works.

If $|S| = k$, we choose the constant a in 7 ways, the remaining $k-1$ elements of S in $\binom{6}{k-1}$ ways, and an image in $S \setminus \{a\}$ for each of the $7-k$ other elements in $(k-1)^{7-k}$ ways. Hence

$$N = 7\sum_{k=1}^{7}\binom{6}{k-1}(k-1)^{7-k} = 7\,(0 + 6 + 240 + 540 + 240 + 30 + 1) = 7 \cdot 1057 = 7399.$$

The remainder when N is divided by 1000 is 399.

12. Let S be the set of all polynomials of the form $z^3 + az^2 + bz + c$, where $a, b,$ and c are integers. Find the number of polynomials in S such that each of its roots z satisfies either $|z| = 20$ or $|z| = 13.$

Solution:

A cubic with real coefficients has either three real roots or one real root and a conjugate pair. The only real numbers with modulus 20 or 13 are ± 20 and $\pm 13,$ so in the all-real case the roots form a multiset of size 3 from those 4 values: $\binom{6}{3} = 20$ polynomials.

Otherwise the roots are $k \in \{\pm 20, \pm 13\}$ and a conjugate pair $r \pm si$ with $s \neq 0$ and $r^2 + s^2 = 400$ or $169.$ Expanding

$$(z-k)\big(z^2 - 2rz + (r^2+s^2)\big)$$

shows the coefficients are $-(2r+k), r^2+s^2+2rk,$ and $-(r^2+s^2)k,$ which are all integers exactly when $2r$ is an integer. On the circle of radius 20 we need $|r| < 20,$ allowing $2r \in \{-39, \ldots, 39\} : 79$ choices; on the circle of radius $13, 2r \in \{-25, \ldots, 25\} : 51$ choices. With 4 choices of $k,$ that gives $4(79+51) = 520$ polynomials, each distinct since the roots determine the polynomial.

In total there are $20 + 520 = 540$ such polynomials.

13. In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Solution:

Let $AB = 2x$ and $AC = BC = y$, so $BD = \frac{y}{4}$, $CD = \frac{3y}{4}$, and $\cos B = \frac{x}{y}$ (drop the altitude from C to the midpoint of $\overline{AB}$). The law of cosines in triangle ABD gives

$$AD^2 = 4x^2 + \frac{y^2}{16} - 2 \cdot 2x \cdot \frac{y}{4} \cdot \frac{x}{y} = 3x^2 + \frac{y^2}{16}.$$

Both CE and BE are medians to $\overline{AD}$, in triangles ACD and ABD respectively. The median formula $4m^2 = 2b^2 + 2c^2 - a^2$ gives

$$28 = 2y^2 + \frac{18y^2}{16} - AD^2 = \frac{49y^2}{16} - 3x^2, \qquad 36 = 8x^2 + \frac{2y^2}{16} - AD^2 = 5x^2 + \frac{y^2}{16}.$$

From the second equation $\frac{y^2}{16} = 36 - 5x^2$; substituting into the first gives $49(36 - 5x^2) - 3x^2 = 28$, so $248x^2 = 1736$, $x^2 = 7$, and then $y^2 = 16$.

The altitude from C has length $\sqrt{y^2 - x^2} = 3$, so the area is $\frac{1}{2} \cdot 2x \cdot 3 = 3\sqrt{7}$, and $m + n = 3 + 7 = 10$.

14. For positive integers n and k, let $f(n,k)$ be the remainder when n is divided by k, and for $n > 1$ let

$$F(n) = \max_{1 \le k \le \frac{n}{2}} f(n,k).$$

Find the remainder when $\sum_{n=20}^{100} F(n)$ is divided by 1000.

Solution:

For $k \le \frac{n}{2}$ the quotient $\lfloor n/k \rfloor$ is at least 2, so the remainder satisfies $f(n,k) \le n - 2k$ as well as $f(n,k) \le k-1$. Write $n = 3m + r$ with $r \in \{0,1,2\}$. Dividing by $k = m+1$ gives quotient 2 and remainder $m + r - 2$, so $F(n) \ge m + r - 2$. Conversely, for $k \ge m+1$, $f(n,k) \le n - 2k \le m + r - 2$, and for smaller k the bound $f(n,k) \le k - 1$ finishes the job: when $r = 2$ it gives at most m for $k \le m+1$; when $r = 1$ it gives at most $m - 1$ for $k \le m$; and when $r = 0$ it gives at most $m - 2$ for $k \le m - 1$, while $k = m$ divides $3m$ exactly, leaving remainder 0. Hence

$$F(3m) = m - 2, \qquad F(3m+1) = m - 1, \qquad F(3m+2) = m.$$

Grouping $n = 20, \ldots, 100$ as triples $3m-1, 3m, 3m+1$ for $m = 7, \ldots, 33$ (note $F(3m-1) = F(3(m-1)+2) = m-1$), each triple contributes $(m-1)+(m-2)+(m-1) = 3m-4$, so

$$\sum_{n=20}^{100} F(n) = \sum_{m=7}^{33} (3m-4) = 3 \cdot \frac{(7+33)\cdot 27}{2} - 4 \cdot 27 = 1620 - 108 = 1512.$$

The requested remainder is 512.

15. Let A, B, C be angles of a triangle with A and C acute and B greater than a right angle satisfying

$$\cos^2 A + \cos^2 B + 2\sin A \sin B \cos C = \frac{15}{8}$$

and

$$\cos^2 B + \cos^2 C + 2\sin B \sin C \cos A = \frac{14}{9}.$$

There are positive integers p, q, r, and s for which

$$\cos^2 C + \cos^2 A + 2\sin C \sin A \cos B = \frac{p - q\sqrt{r}}{s},$$

where $p+q$ and s are relatively prime and r is not divisible by the square of any prime. Find $p+q+r+s$.

Solution:

Replacing each $\cos^2$ by $1-\sin^2$, the first equation becomes $\sin^2 A + \sin^2 B - 2\sin A \sin B \cos C = \frac{1}{8}$. By the law of sines, $\sin A = \frac{a}{2R}$ and so on, so the left side equals

$$\frac{a^2 + b^2 - 2ab\cos C}{4R^2} = \frac{c^2}{4R^2} = \sin^2 C$$

by the law of cosines. Hence $\sin^2 C = 2 - \frac{15}{8} = \frac{1}{8}$. The same argument turns the second equation into $\sin^2 A = 2 - \frac{14}{9} = \frac{4}{9}$, and shows the requested expression equals $2 - \sin^2 B$.

Since A and C are acute, $\cos A = \frac{\sqrt{5}}{3}$ and $\cos C = \frac{\sqrt{14}}{4}$, with $\sin A = \frac{2}{3}$ and $\sin C = \frac{\sqrt{2}}{4}$. Then

$$\sin B = \sin(A+C) = \frac{2}{3}\cdot\frac{\sqrt{14}}{4} + \frac{\sqrt{5}}{3}\cdot\frac{\sqrt{2}}{4} = \frac{2\sqrt{14}+\sqrt{10}}{12},$$

so $\sin^2 B = \frac{66+8\sqrt{35}}{144} = \frac{33+4\sqrt{35}}{72}$.

Therefore $2 - \sin^2 B = \frac{111-4\sqrt{35}}{72}$, and $p+q+r+s = 111+4+35+72 = 222$.

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