2013 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/aime/2013I/solutions

1. The AIME Triathlon consists of a half-mile swim, a 30-mile bicycle ride, and an eight-mile run. Tom swims, bicycles, and runs at constant rates. He runs five times as fast as he swims, and he bicycles twice as fast as he runs. Tom completes the AIME Triathlon in four and a quarter hours. How many minutes does he spend bicycling?

✓

Solution:

Let Tom's swimming speed be s miles per hour. Then he runs at $5s$ and bicycles at $10s$. The total time in hours is

$$\frac{0.5}{s}+\frac{30}{10s}+\frac{8}{5s}=\frac{0.5+3+1.6}{s}=\frac{5.1}{s}=4.25,$$

so $s=\frac{5.1}{4.25}=1.2$ miles per hour.

He bicycles at 12 miles per hour, so the ride takes $\frac{30}{12}=2.5$ hours, which is 150 minutes.

2. Find the number of five-digit positive integers, n, that satisfy the following conditions:

   - the number n is divisible by 5,
   - the first and last digits of n are equal, and
   - the sum of the digits of n is divisible by 5.

✓

Solution:

Since n is divisible by 5, its last digit is 0 or 5; since the first digit equals the last digit and cannot be 0, both are 5. The outer digits contribute 10 to the digit sum, so the three middle digits must also sum to a multiple of 5.

Choose the second and third digits freely, in $10 \cdot 10 = 100$ ways. Whatever their sum is, the fourth digit must land in a prescribed residue class modulo 5, and exactly 2 of the digits 0 through 9 lie in each class. The count is $10 \cdot 10 \cdot 2 = 200$.

3. Let $ABCD$ be a square, and let E and F be points on $\overline{AB}$ and $\overline{BC}$, respectively. The line through E parallel to $\overline{BC}$ and the line through F parallel to $\overline{AB}$ divide $ABCD$ into two squares and two nonsquare rectangles. The sum of the areas of the two squares is $\frac{9}{10}$ of the area of square $ABCD$. Find $\frac{AE}{EB} + \frac{EB}{AE}$.

✓

Solution:

Let $AE = x$ and $EB = y$, so the square has side $x + y$ and the two smaller squares have sides x and y. The condition says

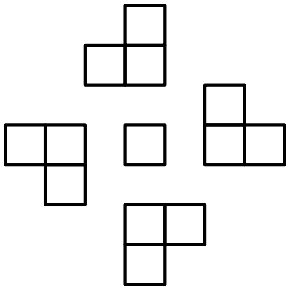
$$x^2 + y^2 = \frac{9}{10}(x + y)^2.$$

Multiplying by 10 and expanding, $10x^2 + 10y^2 = 9x^2 + 18xy + 9y^2$, so $x^2 + y^2 = 18xy$.

Dividing by xy gives

$$\frac{AE}{EB} + \frac{EB}{AE} = \frac{x}{y} + \frac{y}{x} = \frac{x^2 + y^2}{xy} = 18.$$

4. In the array of 13 squares shown below, 8 squares are colored red, and the remaining 5 squares are colored blue. If one of all possible such colorings is chosen at random, the probability that the chosen colored array appears the same when rotated 90° around the central square is $\frac{1}{n}$, where n is a positive integer. Find n.



✓

Solution:

The rotation cycles the four L-shaped arms, so a symmetric coloring colors all four arms identically, and the 12 outer squares contain 4 copies of whatever the arm shows. The number of red squares among the outer twelve is therefore a multiple of 4. Since there are 8 red squares in all, the center must be blue and each arm must contain exactly 2 red squares and 1 blue square.

The blue square within the arm can be chosen in 3 ways, so exactly 3 of the $\binom{13}{5} = 1287$ equally likely colorings are symmetric. The probability is $\frac{3}{1287} = \frac{1}{429}$, so $n = 429$.

5. The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$, where a, b, and c are positive integers. Find $a + b + c$.

✓

Solution:

Rewrite the equation as $9x^3 = x^3 + 3x^2 + 3x + 1 = (x+1)^3$. Taking real cube roots, $\sqrt[3]{9}\,x = x + 1$, so

$$x = \frac{1}{\sqrt[3]{9} - 1}.$$

Multiply numerator and denominator by $\sqrt[3]{81} + \sqrt[3]{9} + 1$; the denominator becomes $(\sqrt[3]{9})^3 - 1 = 8$, so

$$x = \frac{\sqrt[3]{81} + \sqrt[3]{9} + 1}{8}.$$

Thus $a + b + c = 81 + 9 + 8 = 98$.

6. Melinda has three empty boxes and 12 textbooks, three of which are mathematics textbooks. One box will hold any three of her textbooks, one will hold any four of her textbooks, and one will hold any five of her textbooks. If Melinda packs her textbooks into these boxes in random order, the probability that all three mathematics textbooks end up in the same box can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

✓

Solution:

Focus on one box at a time. The box of k books receives a uniformly random k-subset of the 12 books, so the probability that it contains all three math books is $\binom{9}{k-3}/\binom{12}{k}$. For $k = 3, 4, 5$ this gives $\frac{1}{220}$, $\frac{9}{495} = \frac{1}{55}$, and $\frac{36}{792} = \frac{1}{22}$.

The events are disjoint, so the total probability is

$$\frac{1}{220} + \frac{1}{55} + \frac{1}{22} = \frac{1+4+10}{220} = \frac{15}{220} = \frac{3}{44},$$

and $m + n = 3 + 44 = 47$.

7. A rectangular box has width 12 inches, length 16 inches, and height $\frac{m}{n}$ inches, where m and n are relatively prime positive integers. Three faces of the box meet at a corner of the box. The center points of those three faces are the vertices of a triangle with an area of 30 square inches. Find $m + n$.

✓

Solution:

Let the height be h and place the corner at the origin, so the box is $[0, 12] \times [0, 16] \times [0, h]$. The three faces meeting at the origin have centers $P = (6, 8, 0), Q = \left(0, 8, \frac{h}{2}\right)$, and $R = \left(6, 0, \frac{h}{2}\right)$. Then $\overrightarrow{PQ} = \left(-6, 0, \frac{h}{2}\right)$ and $\overrightarrow{PR} = \left(0, -8, \frac{h}{2}\right)$, whose cross product is $(4h, 3h, 48)$. The area is

$$\frac{1}{2}\sqrt{16h^2 + 9h^2 + 48^2} = \frac{1}{2}\sqrt{25h^2 + 2304} = 30,$$

so $25h^2 = 3600 - 2304 = 1296$ and $h = \frac{36}{5}$.

Therefore $m + n = 36 + 5 = 41$.

8. The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m + n$ is divided by 1000.

✓

Solution:

The function is defined when $-1 \le \log_m(nx) \le 1$, that is $\frac{1}{m} \le nx \le m$, so the domain is $\left[\frac{1}{mn}, \frac{m}{n}\right]$, with length

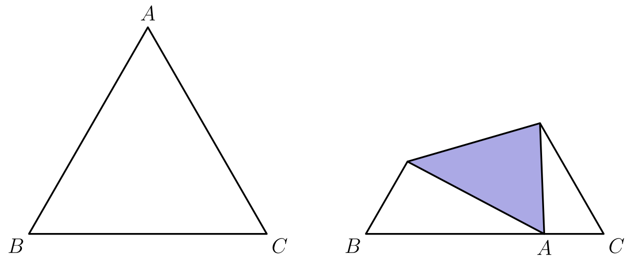
$$\frac{m}{n} - \frac{1}{mn} = \frac{m^2 - 1}{mn} = \frac{1}{2013}.$$

Hence $n = \frac{2013(m^2-1)}{m}$. Since m is relatively prime to $m^2 - 1$, m must divide $2013 = 3 \cdot 11 \cdot 61$.

Because $n \approx 2013m$, the sum $m + n$ grows with m, so take the smallest factor $m = 3$: then $n = \frac{2013 \cdot 8}{3} = 5368$ and $m + n = 5371$.

The remainder upon division by 1000 is 371.

9. A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. The length of the line segment along which the triangle is folded can be written as $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.



✓

Solution:

Let A' be the landing point, with $BA' = 9$ and $CA' = 3$, and let the crease meet $\overline{AB}$ at P and $\overline{AC}$ at Q. Folding preserves distances, so $PA' = PA = x$ and $QA' = QA = y$. In triangle PBA', with $PB = 12 - x$ and $\angle B = 60^\circ$, the law of cosines gives

$$x^2 = (12-x)^2 + 81 - 9(12-x),$$

which simplifies to $15x = 117$, so $x = \frac{39}{5}$. Similarly, in triangle QCA', $y^2 = (12-y)^2 + 9 - 3(12-y)$ gives $21y = 117$, so $y = \frac{39}{7}$.

Finally, in triangle APQ with $\angle A = 60^\circ$,

$$PQ^2 = x^2 + y^2 - xy = 39^2\left(\frac{1}{25} + \frac{1}{49} - \frac{1}{35}\right) = 39^2 \cdot \frac{49+25-35}{1225} = \frac{39^3}{1225},$$

so $PQ = \frac{39\sqrt{39}}{35}$. Thus $m+n+p = 39+35+39 = 113$.

10. There are nonzero integers $a, b, r,$ and s such that the complex number $r + si$ is a zero of the polynomial $P(x) = x^3 - ax^2 + bx - 65.$ For each possible combination of a and $b,$ let $p_{a,b}$ be the sum of the zeros of $P(x).$ Find the sum of the $p_{a,b}$'s for all possible combinations of a and $b.$

✓

Solution:

Since P has real coefficients, $r - si$ is also a zero, and the third zero q is real. The product of the zeros is $q(r^2 + s^2) = 65,$ so q is a nonzero integer and $r^2 + s^2$ is a factor of $65.$ With r, s nonzero, the possibilities are $r^2 + s^2 = 5 = 1^2 + 2^2$ (with $q = 13$), $13 = 2^2 + 3^2$ (with $q = 5$), and $65 = 1^2 + 8^2 = 4^2 + 7^2$ (with $q = 1$).

For each representation $\{u, v\},$ the zero $r + si$ can have $r = \pm u$ or $\pm v,$ giving 4 distinct polynomials (the sign of s changes nothing). The sum of the zeros is $p_{a,b} = q + 2r,$ and over the four choices the $2r$ terms cancel, leaving $4q$ from each representation.

The total is $4 \cdot 13 + 4 \cdot 5 + 4 \cdot 1 + 4 \cdot 1 = 80.$

11. Ms. Math's kindergarten class has 16 registered students. The classroom has a very large number, N, of play blocks which satisfies the conditions:

- If $16, 15,$ or 14 students are present in the class, then in each case all the blocks can be distributed in equal numbers to each student, and

- There are three integers $0 < x < y < z < 14$ such that when $x, y,$ or z students are present and the blocks are distributed in equal numbers to each student, there are exactly three blocks left over.

Find the sum of the distinct prime divisors of the least possible value of N satisfying the above conditions.

✓

Solution:

Divisibility by $16, 15,$ and 14 means $N = 1680m$ where $1680 = \operatorname{lcm}(14, 15, 16) = 2^4 \cdot 3 \cdot 5 \cdot 7.$ Every positive integer less than 14 divides 1680 except $9, 11,$ and $13,$ and a divisor of N leaves remainder $0,$ not $3.$ So necessarily $\{x, y, z\} = \{9, 11, 13\},$ and we need $1680m \equiv 3$ modulo each of $9, 11, 13.$

Since $1680 \equiv 6 \pmod 9,$ the first congruence is $6m \equiv 3 \pmod 9,$ i.e. $m \equiv 2 \pmod 3.$ Since $1680 \equiv 8 \pmod{11},$ we need $8m \equiv 3 \pmod{11},$ i.e. $m \equiv 10 \pmod{11}.$ Since $1680 \equiv 3 \pmod{13},$ we need $m \equiv 1 \pmod{13}.$ By the Chinese remainder theorem these combine to $m \equiv 131 \pmod{429},$ so the least m is $131.$

Then $N = 1680 \cdot 131 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 131,$ and since 131 is prime, the sum of the distinct prime divisors is $2 + 3 + 5 + 7 + 131 = 148.$

12. Let $\triangle PQR$ be a triangle with $\angle P = 75^\circ$ and $\angle Q = 60^\circ$. A regular hexagon $ABCDEF$ with side length 1 is drawn inside $\triangle PQR$ so that side $\overline{AB}$ lies on $\overline{PQ}$, side $\overline{CD}$ lies on $\overline{QR}$, and one of the remaining vertices lies on $\overline{RP}$. There are positive integers $a, b, c,$ and d such that the area of $\triangle PQR$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d.$

✓

Solution:

Note $\angle R = 45^\circ$. Because the hexagon's interior angles are 120°, segments $\overline{BC}$ cut off a corner triangle at Q with two 60° base angles, so triangle BQC is equilateral and $QB = QC = 1$. Put Q at the origin with QR along the positive x-axis. Then $C = (1, 0), D = (2, 0)$, and the hexagon's vertices are $B = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), A = (1, \sqrt{3}), F = (2, \sqrt{3}), E = \left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right).$

Since $\angle R = 45^\circ$, line RP has slope -1. If it passed through E, it would be $x + y = \frac{5+\sqrt{3}}{2}$, which puts F (with $x + y = 2 + \sqrt{3}$) outside the triangle; so the vertex on $\overline{RP}$ is F, and RP is the line $x + y = 2 + \sqrt{3}$. It meets the x-axis at $R = (2 + \sqrt{3},\ 0)$ and the line $y = \sqrt{3}\,x$ (line QP) where $x(1 + \sqrt{3}) = 2 + \sqrt{3}$, giving P height

$$y = \frac{\sqrt{3}(2+\sqrt{3})}{1+\sqrt{3}} = \frac{3+\sqrt{3}}{2}.$$

The area is $\frac{1}{2} \cdot QR \cdot y = \frac{1}{2}(2+\sqrt{3}) \cdot \frac{3+\sqrt{3}}{2} = \frac{9+5\sqrt{3}}{4}$, so $a + b + c + d = 9 + 5 + 3 + 4 = 21.$

13. Triangle AB_0C_0 has side lengths $AB_0 = 12, B_0C_0 = 17,$ and $C_0A = 25.$ For each positive integer $n,$ points B_n and C_n are located on $\overline{AB_{n-1}}$ and $\overline{AC_{n-1}},$ respectively, creating three similar triangles $\triangle AB_nC_n \sim \triangle B_{n-1}C_nC_{n-1} \sim \triangle AB_{n-1}C_{n-1}.$ The area of the union of all triangles $B_{n-1}C_nB_n$ for $n \geq 1$ can be expressed as $\frac{p}{q},$ where p and q are relatively prime positive integers. Find $q.$

✓

Solution:

By Heron's formula with $s = 27,$ the area of $\triangle AB_0C_0$ is $\sqrt{27 \cdot 15 \cdot 10 \cdot 2} = 90.$ In the similarity $\triangle B_0C_1C_0 \sim \triangle AB_0C_0,$ side B_0C_0 corresponds to $AC_0,$ so the ratio is $r = \frac{17}{25},$ and C_1C_0 (corresponding to B_0C_0) equals $17r.$ Hence

$$\frac{AC_1}{AC_0} = \frac{25 - 17r}{25} = 1 - r^2,$$

which is the similarity ratio of $\triangle AB_1C_1$ to $\triangle AB_0C_0.$

Segments $\overline{B_1C_1}$ and $\overline{B_0C_1}$ split $\triangle AB_0C_0$ into the three pieces, so

$$[B_0C_1B_1] = 90\left(1 - r^2 - (1 - r^2)^2\right) = 90\,r^2(1 - r^2).$$

Each successive stage repeats the construction inside $\triangle AB_nC_n,$ scaling all areas by $(1 - r^2)^2,$ and the triangles $B_{n-1}C_nB_n$ have disjoint interiors.

The union's area is the geometric series

$$\frac{90\,r^2(1 - r^2)}{1 - (1 - r^2)^2} = \frac{90(1 - r^2)}{2 - r^2} = 90 \cdot \frac{336/625}{961/625} = \frac{90 \cdot 336}{961}.$$

Since $961 = 31^2$ shares no factor with $90 \cdot 336 = 30240,$ the answer is $q = 961.$

14. For $\pi \leq \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \dots$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \dots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Then $\sin\theta = -\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

✓

Solution:

The signs and the alternation between sines and cosines suggest powers of i : indeed

$$Q + iP = 1 + \frac{1}{2}ie^{i\theta} + \frac{1}{4}i^2e^{2i\theta} + \frac{1}{8}i^3e^{3i\theta} + \dots = \frac{1}{1 - \frac{ie^{i\theta}}{2}} = \frac{2}{2 - ie^{i\theta}}.$$

Since $2 - ie^{i\theta} = (2 + \sin\theta) - i\cos\theta$, multiplying by the conjugate gives

$$Q + iP = \frac{2(2 + \sin\theta) + 2i\cos\theta}{5 + 4\sin\theta},$$

so $\frac{P}{Q} = \frac{\cos\theta}{2+\sin\theta}$.

Setting $\frac{\cos\theta}{2+\sin\theta} = \frac{2\sqrt{2}}{7}$ and squaring, $49(1 - \sin^2\theta) = 8(2 + \sin\theta)^2$, which rearranges to

$$57\sin^2\theta + 32\sin\theta - 17 = (3\sin\theta - 1)(19\sin\theta + 17) = 0.$$

Since $\pi \leq \theta < 2\pi$ forces $\sin\theta \leq 0$, we get $\sin\theta = -\frac{17}{19}$ (and then $\cos\theta = \frac{6\sqrt{2}}{19} > 0$, consistent with the positive ratio). Thus $m + n = 17 + 19 = 36$.

15. Let N be the number of ordered triples (A, B, C) of integers satisfying the conditions

- $0 \leq A < B < C \leq 99$,
- there exist integers a, b, and c, and prime p where $0 \leq b < a < c < p$,
- p divides $A - a, B - b$, and $C - c$, and
- each ordered triple (A, B, C) and each ordered triple (b, a, c) form arithmetic sequences.

Find N.

✓

Solution:

Let d be the common difference of (b, a, c), so $a - b = c - a = d > 0$ and $c = b + 2d < p$, whence $0 < 2d < p$. Let $D > 0$ be the common difference of (A, B, C). Reducing mod p, we get $D = B - A \equiv b - a = -d$ and $D = C - B \equiv c - b = 2d$, so $p \mid 3d$. Since $0 < d < p$, the prime p cannot divide d, so $p = 3$; then $2d < 3$ gives $d = 1$ and $(b, a, c) = (0, 1, 2)$.

So the valid triples are exactly the increasing arithmetic progressions in $[0, 99]$ with $A \equiv 1, B \equiv 0$, $C \equiv 2 \pmod 3$. Write $A = 1 + 3j$ with $j \geq 0$; the difference satisfies $D \equiv -1 \equiv 2 \pmod 3$, so $D = 2 + 3k$ with $k \geq 0$. The constraint is $C = A + 2D = 5 + 3j + 6k \leq 99$, i.e. $j + 2k \leq 31$, and every such pair (j, k) works.

For each $k = 0, 1, \ldots, 15$ there are $32 - 2k$ choices of j, so

$$N = \sum_{k=0}^{15} (32 - 2k) = 16 \cdot 32 - 2 \cdot \frac{15 \cdot 16}{2} = 512 - 240 = 272.$$

Problems: https://live.poshenloh.com/past-contests/aime/2013I