2012 AIME II Solutions

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https://live.poshenloh.com/past-contests/aime/2012II/solutions

1. Find the number of ordered pairs of positive integer solutions (m, n) to the equation

$$20m + 12n = 2012.$$

Solution:

Dividing by 4 gives $5m + 3n = 503$. Reducing modulo 3, we need $2m \equiv 503 \equiv 2 \pmod 3$, so $m \equiv 1 \pmod 3$. Write $m = 3k + 1$ with $k \geq 0$; then $3n = 503 - 5(3k + 1) = 498 - 15k$, so $n = 166 - 5k$.

This is positive exactly when $5k \leq 165$, that is $k \leq 33$. So $k = 0, 1, \ldots, 33$ all work, giving 34 ordered pairs.

2. Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27, b_1 = 99,$ and $a_{15} = b_{11}$. Find a_9.

Solution:

Let r be the shared common ratio. Then $a_{15} = 27r^{14}$ and $b_{11} = 99r^{10}$, so $27r^{14} = 99r^{10}$ gives $r^4 = \frac{99}{27} = \frac{11}{3}$.

Therefore

$$a_9 = 27r^8 = 27\left(\frac{11}{3}\right)^2 = 27 \cdot \frac{121}{9} = 3 \cdot 121 = 363.$$

3. At a certain university, the division of mathematical sciences consists of the departments of mathematics, statistics, and computer science. There are two male and two female professors in each department. A committee of six professors is to contain three men and three women and must also contain two professors from each of the three departments. Find the number of possible committees that can be formed subject to these requirements.

Solution:

Each department contributes exactly two committee members. If every department sends one man and one woman, there are $2 \cdot 2 = 4$ choices per department, for $4^3 = 64$ committees.

Otherwise some department sends two men. To keep three of each gender, another department must then send its two women, and the remaining department sends one man and one woman. There are 3 ways to pick the all-male department, 2 ways to pick the all-female department, and $2 \cdot 2 = 4$ choices in the mixed department (the two-man and two-woman selections are forced), for $3 \cdot 2 \cdot 4 = 24$ committees.

The total is $64 + 24 = 88.$

4. Ana, Bob, and Cao bike at constant rates of 8.6 meters per second, 6.2 meters per second, and 5 meters per second, respectively. They all begin biking at the same time from the northeast corner of a rectangular field whose longer side runs due west. Ana starts biking along the edge of the field, initially heading west, Bob starts biking along the edge of the field, initially heading south, and Cao bikes in a straight line across the field to a point D on the south edge of the field. Cao arrives at point D at the same time that Ana and Bob arrive at D for the first time. The ratio of the field's length to the field's width to the distance from point D to the southeast corner of the field can be represented as $p : q : r$, where $p, q,$ and r are positive integers with p and q relatively prime. Find $p + q + r$.

Solution:

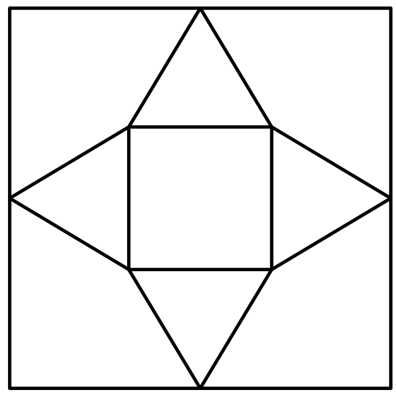
Let the field have length L (west) and width W (south) with $L > W$, and let x be the distance from D to the southeast corner. Ana rides around the perimeter a distance $2L + W - x$, Bob rides $W + x$, and Cao rides $\sqrt{W^2 + x^2}$, all in the same time:

$$\frac{2L + W - x}{8.6} = \frac{W + x}{6.2} = \frac{\sqrt{W^2 + x^2}}{5}.$$

The first equality gives $L = \frac{6W+37x}{31}$. Squaring the second, $25(W + x)^2 = 38.44\,(W^2 + x^2)$, which simplifies to $168W^2 - 625Wx + 168x^2 = 0$, factoring as $(24W - 7x)(7W - 24x) = 0$.

The root $x = \frac{7W}{24}$ gives $L = \frac{13W}{24} < W$, which is impossible, so $x = \frac{24W}{7}$ and then $L = \frac{30W}{7}$. The ratio is $L : W : x = 30 : 7 : 24$, and $p + q + r = 30 + 7 + 24 = 61.$

5. In the accompanying figure, the outer square S has side length 40. A second square S' of side length 15 is constructed inside S with the same center as S and with sides parallel to those of S. From each midpoint of a side of S, segments are drawn to the two closest vertices of S'. The result is a four-pointed starlike figure inscribed in S. The star figure is cut out and then folded to form a pyramid with base S'. Find the volume of this pyramid.



Solution:

Folding the star along the sides of S' lifts the four triangular points so that their tips (the midpoints of the sides of S) meet at a single apex V. Let M be the center of S' and P the midpoint of one of its sides. In the flat figure, the distance from P to the tip of its triangle is $20 - \frac{15}{2} = \frac{25}{2}$, and this becomes the slant PV after folding.

Triangle PMV has a right angle at M, with $PM = \frac{15}{2}$, so the height is

$$VM = \sqrt{\left(\frac{25}{2}\right)^2 - \left(\frac{15}{2}\right)^2} = \sqrt{100} = 10.$$

The volume is $\frac{1}{3} \cdot 15^2 \cdot 10 = 750.$

6. Let $z = a + bi$ be the complex number with $|z| = 5$ and $b > 0$ such that the distance between $(1+2i)z^3$ and z^5 is maximized, and let $z^4 = c + di$. Find $c + d$.

Solution:

The distance is $|(1+2i)z^3 - z^5| = |z|^3 \cdot |1 + 2i - z^2| = 125\,|1 + 2i - z^2|$. As z runs over the circle $|z| = 5$ with $b > 0$, the square z^2 attains every point of the circle $|w| = 25$ (the condition $b > 0$ merely selects one of the two square roots). The point of that circle farthest from $1 + 2i$ is diametrically opposite in direction:

$$z^2 = -25 \cdot \frac{1+2i}{|1+2i|} = -5\sqrt{5}\,(1+2i).$$

Squaring, $z^4 = 125\,(1+2i)^2 = 125\,(-3+4i) = -375 + 500i$, so $c + d = -375 + 500 = 125.$

7. Let S be the increasing sequence of positive integers whose binary representation has exactly 8 ones. Let N be the 1000th number in S. Find the remainder when N is divided by 1000.

Solution:

There are $\binom{12}{8} = 495$ members of S below 2^{12} and $\binom{13}{8} = 1287$ below 2^{13}, so N has 13 binary digits, and exactly $1287 - 1000 = 287$ members below 2^{13} exceed it. The $\binom{11}{6} = 462$ members whose binary representations begin 11 are the largest ones, so N begins with 11 and is the $462 - 287 = 175$th smallest of them.

Among these, $\binom{9}{6} = 84$ begin 1100, the next $\binom{8}{5} = 56$ begin 11010, and the next $\binom{7}{4} = 35$ begin 110110. Since $84 + 56 + 35 = 175$, the number N is the largest member beginning 110110, namely 1101101111000 in binary.

Its value is $2^{12} + 2^{11} + 2^9 + 2^8 + 2^6 + 2^5 + 2^4 + 2^3 = 7032$, so the remainder upon division by 1000 is 32.

8. The complex numbers z and w satisfy the system

$$z + \frac{20i}{w} = 5 + i, \qquad w + \frac{12i}{z} = -4 + 10i.$$

Find the smallest possible value of $|zw|^2$.

Solution:

Multiplying the two equations gives

$$zw + 12i + 20i - \frac{240}{zw} = (5+i)(-4+10i) = -30 + 46i,$$

so $zw - \frac{240}{zw} = -30 + 14i$. Setting $v = zw$ yields $v^2 + (30 - 14i)v - 240 = 0$.

By the quadratic formula, $v = -15 + 7i \pm \sqrt{(15 - 7i)^2 + 240} = -15 + 7i \pm \sqrt{416 - 210i}$. Writing $(a + bi)^2 = 416 - 210i$ requires $a^2 - b^2 = 416$ and $ab = -105$, which gives $a + bi = \pm(21 - 5i)$. Hence $v = 6 + 2i$ or $v = -36 + 12i$, with $|v|^2 = 40$ or 1440.

The smaller value is attained: $z = 1 - i, w = 2 + 4i$ satisfies both equations with $zw = 6 + 2i$. So the smallest possible value of $|zw|^2$ is 40.

9. Let x and y be real numbers such that $\frac{\sin x}{\sin y} = 3$ and $\frac{\cos x}{\cos y} = \frac{1}{2}$. The value of $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:

From the double-angle formula,

$$\frac{\sin 2x}{\sin 2y} = \frac{2\sin x \cos x}{2\sin y \cos y} = 3 \cdot \frac{1}{2} = \frac{3}{2}.$$

Squaring the given equations, $\sin^2 x = 9\sin^2 y$ and $\cos^2 x = \frac{1}{4}\cos^2 y$. Adding, $1 = 9(1 - \cos^2 y) + \frac{1}{4}\cos^2 y$, so $\frac{35}{4}\cos^2 y = 8$ and $\cos^2 y = \frac{32}{35}$. Then $\cos 2y = 2\cos^2 y - 1 = \frac{29}{35}$ and $\cos 2x = 2\cos^2 x - 1 = \frac{1}{2}\cos^2 y - 1 = -\frac{19}{35}$, so $\frac{\cos 2x}{\cos 2y} = -\frac{19}{29}$.

The requested value is $\frac{3}{2} - \frac{19}{29} = \frac{87-38}{58} = \frac{49}{58}$, and $p + q = 49 + 58 = 107$.

10. Find the number of positive integers n less than 1000 for which there exists a positive real number x such that $n = x\lfloor x \rfloor$.

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Solution:

Fix $\lfloor x \rfloor = a \geq 1$. For $a \leq x < a + 1$ the product $n = ax$ ranges over $[a^2, a^2 + a)$, and every integer n in that interval is achieved by $x = \frac{n}{a}$ (which indeed has floor a). So each a contributes exactly a values of n, namely $a^2, a^2 + 1, \ldots, a^2 + a - 1$.

For $a = 31$ the largest value is $31^2 + 30 = 991 < 1000$, so all values through $a = 31$ qualify, while $a = 32$ already starts at 1024. The count is $1 + 2 + \cdots + 31 = \frac{31 \cdot 32}{2} = 496$.

11. Let $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$, and for $n \geq 2$, define $f_n(x) = f_1(f_{n-1}(x))$. The value of x that satisfies $f_{1001}(x) = x - 3$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Combining fractions, $f_1(x) = \frac{2(3x+1)-9}{3(3x+1)} = \frac{6x-7}{9x+3}$. Composing once, $f_2(x) = \frac{6f_1(x)-7}{9f_1(x)+3} = \frac{-3x-7}{9x-6}$, and composing again gives $f_3(x) = x$.

So the iteration is periodic with period 3. Since $1001 \equiv 2 \pmod 3$, we have $f_{1001} = f_2$, and the equation becomes

$$\frac{-3x-7}{9x-6} = x - 3,$$

that is $9x^2 - 33x + 18 = -3x - 7$, or $9x^2 - 30x + 25 = (3x-5)^2 = 0$.

The unique solution is $x = \frac{5}{3}$, so $m + n = 5 + 3 = 8$.

12. For a positive integer p, define the positive integer n to be p-*safe* if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is $\{3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 23, \ldots\}$. Find the number of positive integers less than or equal to $10{,}000$ which are simultaneously 7-safe, 11-safe, and 13 -safe.

Solution:

Being p-safe depends only on $n \bmod p$: it requires $3 \le n \bmod p \le p-3$. That allows 2 residues modulo 7, 6 residues modulo 11, and 8 residues modulo 13. Since $7 \cdot 11 \cdot 13 = 1001$, the Chinese remainder theorem gives exactly $2 \cdot 6 \cdot 8 = 96$ safe residues modulo 1001, so each block of 1001 consecutive integers contains 96 safe numbers.

The integers 1 through 10010 form ten such blocks, containing 960 safe numbers. It remains to discard the safe numbers among $10001, \ldots, 10010$. Their residues modulo 7 run $5, 6, 0, 1, 2, 3, 4, 5, 6, 0$, so only 10006 and 10007 are 7-safe; both are also 11-safe (residues 7 and 8) and 13-safe (residues 9 and 10).

Therefore the count is $960 - 2 = 958$.

13. Equilateral $\triangle ABC$ has side length $\sqrt{111}$. There are four distinct triangles AD_1E_1, AD_1E_2, AD_2E_3, and AD_2E_4, each congruent to $\triangle ABC$, with $BD_1 = BD_2 = \sqrt{11}$. Find $\sum_{k=1}^{4}(CE_k)^2$.

Solution:

Write $s = \sqrt{111}$ and $r = \sqrt{11}$. Since each triangle AD_iE_k is congruent to $\triangle ABC$, we have $AD_i = AE_k = s$, so D_1 and D_2 are the two intersections of the circle of radius s about A with the circle of radius r about B; they are mirror images across line AB, so $\angle BAD_1 = \angle BAD_2 = \theta$ with D_1 and D_2 on opposite sides of AB. Each E_k is the image of its D_i rotated $\pm 60^\circ$ about A. Measuring signed angles from ray AB, with C at $+60^\circ$, the rays AD_i sit at $\pm\theta$ and the rays AE_k at $\pm\theta \pm 60^\circ$, so the four angles $\angle CAE_k$ are $\theta, \theta, 120^\circ - \theta$, and $120^\circ + \theta$.

Since $AC = AE_k = s$, the law of cosines gives $(CE_k)^2 = 2s^2(1 - \cos \angle CAE_k)$. Using $\cos(120^\circ - \theta) + \cos(120^\circ + \theta) = 2\cos 120^\circ \cos\theta = -\cos\theta$, the four angles' cosines sum to $2\cos\theta - \cos\theta = \cos\theta$, so

$$\sum_{k=1}^{4}(CE_k)^2 = 2s^2(4 - \cos\theta).$$

Applying the law of cosines in triangle ABD_1 (with $AB = AD_1 = s$) gives $r^2 = 2s^2(1 - \cos\theta)$, so $2s^2\cos\theta = 2s^2 - r^2$. Therefore the sum equals $8s^2 - (2s^2 - r^2) = 6s^2 + r^2 = 6 \cdot 111 + 11 = 677$.

14. In a group of nine people each person shakes hands with exactly two of the other people from the group. Let N be the number of ways this handshaking can occur. Consider two handshaking arrangements different if and only if at least two people who shake hands under one arrangement do not shake hands under the other arrangement. Find the remainder when N is divided by 1000.

Solution:

An arrangement in which everyone shakes exactly two hands is a disjoint union of cycles of length at least 3 covering all nine people. The possible cycle-length partitions of 9 are $3+3+3, 3+6, 4+5$, and 9. On k given people, the number of distinct cycles is $\frac{(k-1)!}{2}$.

For $3+3+3:$ split into three unordered triples in $\frac{1}{3!}\binom{9}{3}\binom{6}{3} = 280$ ways, one cycle each: 280. For $3+6: \binom{9}{3} \cdot 1 \cdot \frac{5!}{2} = 84 \cdot 60 = 5040$. For $4+5: \binom{9}{4} \cdot \frac{3!}{2} \cdot \frac{4!}{2} = 126 \cdot 3 \cdot 12 = 4536$. For a single 9-cycle: $\frac{8!}{2} = 20160$.

In total $N = 280 + 5040 + 4536 + 20160 = 30016$, so the remainder modulo 1000 is 16.

15. Triangle ABC is inscribed in circle ω with $AB = 5, BC = 7,$ and $AC = 3.$ The bisector of angle A meets side $\overline{BC}$ at D and circle ω at a second point $E.$ Let γ be the circle with diameter $\overline{DE}.$ Circles ω and γ meet at E and a second point $F.$ Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n.$

Solution:

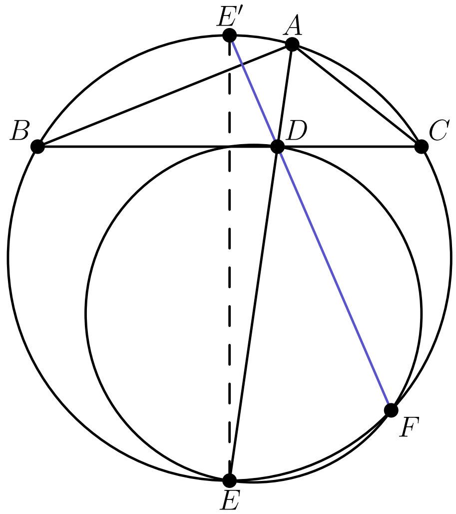
Let E' be the point of ω diametrically opposite $E.$ Since $\overline{DE}$ is a diameter of γ, the angle $\angle DFE = 90^\circ$, and since $\overline{EE'}$ is a diameter of ω, also $\angle E'FE = 90^\circ.$ Both FD and FE' are perpendicular to FE, so D lies on line $E'F$: the point F is the second intersection of line $E'D$ with $\omega.$

Set $B = (0, 0)$ and $C = (7, 0)$; then $A = \left(\frac{65}{14}, \frac{15\sqrt{3}}{14}\right)$. The bisector gives $\frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{3}$, so $D = \left(\frac{35}{8}, 0\right)$. Since E is the midpoint of arc BC not containing A, both E and E' lie on the vertical line $x = \frac{7}{2}$ through the center $O = \left(\frac{7}{2}, -\frac{7\sqrt{3}}{6}\right)$, which satisfies $|OB| = |OA|$ with $R^2 = \frac{49}{3}$. Thus $E = \left(\frac{7}{2}, -\frac{7\sqrt{3}}{2}\right)$ and $E' = \left(\frac{7}{2}, \frac{7\sqrt{3}}{6}\right)$.

The direction from E' to D is proportional to $(3, -4\sqrt{3})$, and the point $E' + t\,(3, -4\sqrt{3})$ lies on ω when $57t^2 - 56t = 0$, so $t = \frac{56}{57}$ gives $F = \left(\frac{245}{38}, -\frac{105\sqrt{3}}{38}\right)$. Then

$$AF^2 = \left(\frac{240}{133}\right)^2 + 3\left(\frac{510}{133}\right)^2 = \frac{837900}{17689} = \frac{900}{19},$$

so $m + n = 900 + 19 = 919.$



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