2012 AIME I Solutions

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1. Find the number of positive integers with three not necessarily distinct digits, abc, with $a \neq 0$ and $c \neq 0$ such that both abc and cba are multiples of 4.

Solution:

An integer is a multiple of 4 exactly when its last two digits form a multiple of 4, so we need $4 \mid 10b + c$ and $4 \mid 10b + a$. In particular a and c are even, and subtracting the two conditions shows $4 \mid a - c$. The even nonzero digits split by remainder mod 4 into $\{2, 6\}$ and $\{4, 8\}$, so a and c must both come from the same one of these sets: 4 ordered pairs (a, c) from each.

If $c \equiv 2 \pmod 4$, then $10b + c \equiv 2b + 2 \pmod 4$ requires b odd (5 choices), and the condition on $10b + a$ holds automatically since $a \equiv c \pmod 4$. If $c \equiv 0 \pmod 4$, then b must be even (5 choices).

The count is $4 \cdot 5 + 4 \cdot 5 = 40$.

2. The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

Solution:

If the sequence has n terms, the amounts added are the first n odd numbers, whose sum is n^2. Thus $n^2 = 836 - 715 = 121$, so $n = 11$.

The average of the 11 terms is $\frac{715}{11} = 65$, which equals the middle (sixth) term of the arithmetic sequence. The first and last terms also average to 65, so they add to 130.

The requested sum is $65 + 130 = 195$.

3. Nine people sit down for dinner where there are three choices of meals. Three people order the beef meal, three order the chicken meal, and three order the fish meal. The waiter serves the nine meals in random order. Find the number of ways in which the waiter could serve the meal types to the nine people so that exactly one person receives the type of meal ordered by that person.

Solution:

Choose the one person served correctly (9 ways); by symmetry say they ordered beef. The remaining meals — 2 beef, 3 chicken, and 3 fish — must go to the other 8 people (2 beef, 3 chicken, and 3 fish orderers) with nobody matched. Track where the 2 leftover beef meals go: to chicken or fish orderers.

If both go to the same group, say to two of the three chicken orderers ($3+3=6$ ways counting both groups), then the third chicken orderer must receive fish, the three fish orderers must take the three chicken meals, and the two beef orderers take the remaining fish: everything is forced. If one goes to a chicken orderer and one to a fish orderer ($3 \cdot 3 = 9$ ways), the other two chicken orderers must take fish and the other two fish orderers must take chicken, leaving one chicken and one fish meal to split between the two beef orderers (2 ways).

The total is $9\,(6 + 9 \cdot 2) = 216.$

4. Butch and Sundance need to get out of Dodge. To travel as quickly as possible, each alternates walking and riding their only horse, Sparky, as follows. Butch begins by walking while Sundance rides. When Sundance reaches the first of the hitching posts that are conveniently located at one-mile intervals along their route, he ties Sparky to the post and begins walking. When Butch reaches Sparky, he rides until he passes Sundance, then leaves Sparky at the next hitching post and resumes walking, and they continue in this manner. Sparky, Butch, and Sundance walk at $6, 4,$ and 2.5 miles per hour, respectively. The first time Butch and Sundance meet at a milepost, they are n miles from Dodge, and they have been traveling for t minutes. Find $n + t$.

Solution:

Walking a mile takes Sparky 10 minutes, Butch 15, and Sundance 24. The horse advances along the same route as the men and is ridden over each mile by exactly one of them, so if Butch walks x of the n miles and rides the other $n - x$, then Sundance rides those x miles and walks the remaining $n - x$.

When they meet at a milepost they have been traveling for the same amount of time, so

$$15x + 10(n - x) = 10x + 24(n - x),$$

which simplifies to $19x = 14n$. Since the handoffs happen at mileposts, x and n are integers, and the smallest positive solution is $x = 14, n = 19$.

Then $t = 15 \cdot 14 + 10 \cdot 5 = 260$ minutes, so $n + t = 19 + 260 = 279$.

5. Let B be the set of all binary integers that can be written using exactly 5 zeros and 8 ones where leading zeros are allowed. If all possible subtractions are performed in which one element of B is subtracted from another, find the number of times the answer 1 is obtained.

Solution:

We must count pairs of elements of B differing by 1, say m and $m+1$. Adding 1 to a binary number turns its trailing block $011\cdots1$ (a zero followed by k ones) into $100\cdots0$, changing the number of ones by $1-k$. Both numbers have exactly eight ones precisely when $k=1: m$ ends in 01, $m+1$ ends in 10, and the two numbers agree everywhere else.

The shared first eleven digits then consist of the remaining seven ones and four zeros, and since leading zeros are allowed, every arrangement gives a valid pair: $\binom{11}{4}=330.$ Each pair produces the answer 1 exactly once, so the count is $330.$

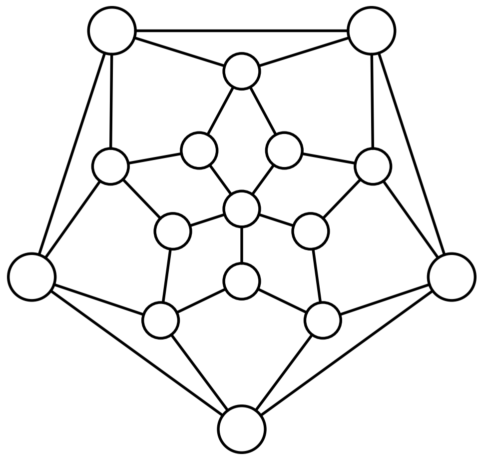
6. The complex numbers z and w satisfy $z^{13}=w, w^{11}=z$, and the imaginary part of z is $\sin\left(\frac{m\pi}{n}\right)$ for relatively prime positive integers m and n with $m<n$. Find n.

Solution:

Substituting, $z=w^{11}=(z^{13})^{11}=z^{143}$, and $z\neq 0$, so $z^{142}=1.$ Conversely, any 142 nd root of unity z works with $w=z^{13}$, since then $w^{11}=z^{143}=z.$

Hence $z=\cos\frac{2k\pi}{142}+i\sin\frac{2k\pi}{142}$ for some integer k, and the imaginary part of z is $\sin\frac{k\pi}{71}$. Since 71 is prime, for every k with $1\leq k\leq 70$ the fraction $\frac{k}{71}$ is already in lowest terms, matching the required form $\sin\left(\frac{m\pi}{n}\right)$ with $m<n$. Thus $n=71.$

7. At each of the sixteen circles in the network below stands a student. A total of 3360 coins are distributed among the sixteen students. All at once, all students give away all their coins by passing an equal number of coins to each of their neighbors in the network. After the trade, all students have the same number of coins as they started with. Find the number of coins the student standing at the center circle had originally.



Solution:

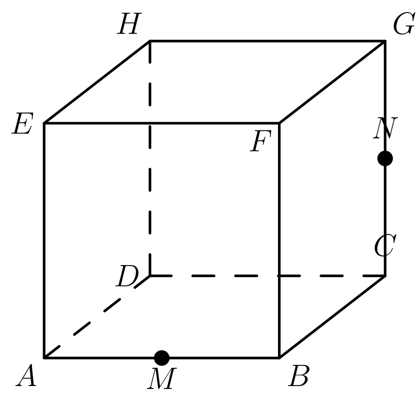
Group the sixteen circles into rings: the center, the inner ring of five, the middle ring of five, and the outer ring of five, holding p, q, r, and s coins in total, respectively. The center has 5 neighbors (the inner ring); each inner student has 3 (the center and two middle students); each middle student has 4 (two inner and two outer); each outer student has 4 (two middle and two outer). A student with k neighbors sends $\frac{1}{k}$ of their coins to each neighbor.

Summing the trades over each ring (for example, the outer ring receives a quarter of each middle student's coins twice over, which totals $\frac{r}{2}$) gives

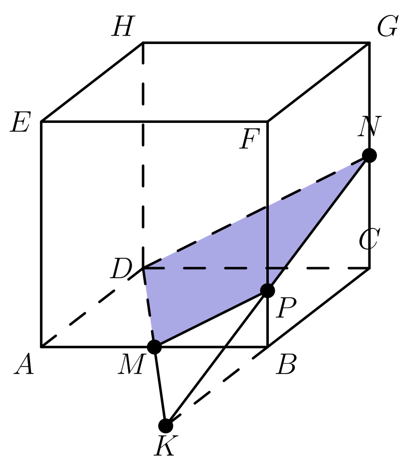
$$p = \frac{q}{3}, \qquad q = p + \frac{r}{2}, \qquad r = \frac{2q}{3} + \frac{s}{2}, \qquad s = \frac{r}{2} + \frac{s}{2}.$$

The first equation gives $q = 3p$, the second then gives $r = 2(q - p) = 4p$, and the last gives $s = r = 4p$. The total is $p + 3p + 4p + 4p = 12p = 3360$, so the center student had $p = 280$ coins.

8. Cube $ABCDEFGH$, labeled as shown below, has edge length 1 and is cut by a plane passing through vertex D and the midpoints M and N of $\overline{AB}$ and $\overline{CG}$, respectively. The plane divides the cube into two solids. The volume of the larger of the two solids can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.



Solution:



Extend the cutting plane. In the bottom face, line DM meets line CB extended beyond B at a point K; since $MB \parallel DC$ and $MB = \frac{1}{2}DC$, segment MB is a midline of triangle KDC, so B is the midpoint of $\overline{CK}$ and $CK = 2$. The plane also cuts edge BF at a point P, and the piece of the cube cut off past the plane is the pyramid $KDCN$ with the small pyramid $KMBP$ sliced away.

Pyramid $KDCN$ has base DCN, a right triangle with legs $DC = 1$ and $CN = \frac{1}{2}$, and its apex K is at distance $CK = 2$ from the plane of that base, so its volume is $\frac{1}{3} \cdot$

$\frac{1}{4} \cdot 2 = \frac{1}{6}$. Pyramid $KMBP$ is similar to $KDCN$ with ratio $\frac{KB}{KC} = \frac{1}{2}$, so its volume is $\frac{1}{8} \cdot \frac{1}{6} = \frac{1}{48}$.

The smaller piece therefore has volume $\frac{1}{6} - \frac{1}{48} = \frac{7}{48}$, and the larger piece has volume $1 - \frac{7}{48} = \frac{41}{48}$, giving $p + q = 41 + 48 = 89$.

9. Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:

Write $x = 2^a, y = 2^b, z = 2^c$. Then $\log_x(2y) = \frac{b+1}{a}$, $\log_{2x}(4z) = \frac{c+2}{a+1}$, and $\log_{2x^4}(8yz) = \frac{b+c+3}{4a+1}$, so the condition is

$$\frac{2(b+1)}{a} = \frac{2(c+2)}{a+1} = \frac{b+c+3}{4a+1} \neq 0.$$

From the first two, $\frac{b+1}{a} = \frac{c+2}{a+1}$, and equal ratios also equal their mediant $\frac{b+c+3}{2a+1}$. Comparing with the third expression gives $\frac{2(b+c+3)}{2a+1} = \frac{b+c+3}{4a+1}$. The common value is nonzero, so $b + c + 3 \neq 0$, and thus $2(4a + 1) = 2a + 1$, giving $a = -\frac{1}{6}$. Then $\frac{b+1}{-1/6} = \frac{c+2}{5/6}$ yields $c + 2 = -5(b + 1)$, that is, $5b + c = -7$.

Therefore $xy^5z = 2^{a+5b+c} = 2^{-1/6-7} = \frac{1}{2^{43/6}}$, so $p + q = 43 + 6 = 49$.

10. Let $\mathcal{S}$ be the set of all perfect squares whose rightmost three digits in base 10 are $256.$ Let $\mathcal{T}$ be the set of all numbers of the form $\frac{x-256}{1000}$, where x is in $\mathcal{S}.$ In other words, $\mathcal{T}$ is the set of numbers that result when the last three digits of each number in $\mathcal{S}$ are truncated. Find the remainder when the tenth smallest element of $\mathcal{T}$ is divided by $1000.$

Solution:

A square n^2 ends in 256 exactly when $1000 \mid n^2 - 256 = (n-16)(n+16).$ Modulo $8 : n^2 \equiv 0 \pmod 8$ forces $4 \mid n.$ Modulo 125 : the factors $n \pm 16$ differ by $32,$ so 5 divides at most one of them, and hence 125 must divide a single factor: $n \equiv \pm 16 \pmod{125}.$ Because 16 is a multiple of $4,$ the two conditions combine to $n \equiv \pm 16 \pmod{500}.$

So $\mathcal{S}$ consists of the numbers $(500m \pm 16)^2,$ whose square roots in increasing order are $16, 484, 516, 984, 1016, \ldots.$ The tenth smallest element of $\mathcal{S}$ is $(500 \cdot 5 - 16)^2 = 2484^2.$

The corresponding element of $\mathcal{T}$ is $\frac{2484^2-256}{1000} = \frac{2468 \cdot 2500}{1000} = 6170,$ whose remainder upon division by 1000 is $170.$

11. A frog begins at $P_0 = (0, 0)$ and makes a sequence of jumps according to the following rule: from $P_n = (x_n, y_n)$, the frog jumps to P_{n+1}, which may be any of the points $(x_n + 7, y_n + 2)$, $(x_n + 2, y_n + 7)$, $(x_n - 5, y_n - 10)$, or $(x_n - 10, y_n - 5)$. There are M points (x, y) with $|x| + |y| \leq 100$ that can be reached by a sequence of such jumps. Find the remainder when M is divided by 1000.

Solution:

Each jump changes $x + y$ by $+9$ or -15 and changes $x - y$ by ± 5. Starting from $(0, 0)$, every reachable point therefore has $x + y = 3j$ and $x - y = 5k$ for integers j and k; moreover $x = \frac{3j+5k}{2}$ must be an integer, so j and k have the same parity. Since $|x| + |y| = \max(|x + y|,\ |x - y|)$, the condition $|x| + |y| \leq 100$ becomes $|j| \leq 33$ and $|k| \leq 20$.

Conversely, every such point is reachable: a single jump moves between neighboring lines $x - y = 5k$ (changing j by 3 or -5, which flips its parity), and two-jump combinations translate by $(9, 9)$ or $(-15, -15)$, which combine — two of the former plus one of the latter — into the shift $(3, 3)$, moving j by 2 along a fixed line. Together these reach every pair (j, k) of equal parity.

Counting: even j (33 values) pairs with even k (21 values), and odd j (34 values) with odd k (20 values), so $M = 33 \cdot 21 + 34 \cdot 20 = 1373$. The remainder is 373.

12. Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m+n+p$.

Solution:

The trisectors make $\angle ACD = \angle DCE = \angle ECB = 30^\circ$. In triangle DCB, ray CE bisects the 60° angle DCB, so the angle bisector theorem gives $\frac{CD}{CB} = \frac{DE}{EB} = \frac{8}{15}$. Scale the triangle so that $CD = 8$ and $CB = 15$.

By the Law of Cosines in triangle DCB,

$$BD^2 = 8^2 + 15^2 - 2 \cdot 8 \cdot 15 \cos 60^\circ = 169,$$

so $BD = 13$. Applying the Law of Cosines again in the same triangle, $8^2 = 13^2 + 15^2 - 2 \cdot 13 \cdot 15 \cos B$, which gives $\cos B = \frac{11}{13}$.

Then $\sin B = \sqrt{1 - \frac{121}{169}} = \frac{4\sqrt{3}}{13}$, so $\tan B = \frac{4\sqrt{3}}{11}$ and $m+n+p = 4+11+3 = 18$.

13. Three concentric circles have radii $3, 4,$ and 5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as $a + \frac{b}{c}\sqrt{d}$, where $a, b, c,$ and d are positive integers, b and c are relatively prime, and d is not divisible by the square of any prime. Find $a + b + c + d$.

Solution:

Let O be the common center and label the triangle ABC with $OA = 3, OB = 4,$ $OC = 5$. Rotate the plane by 60° about A so that B maps to C, and let P be the image of O. Then triangle AOP is equilateral, so $OP = OA = 3$, and PC, the image of OB, has length 4.

Triangle OPC has sides $3, 4, 5$, so $\angle OPC = 90^\circ$. In the configuration giving the largest triangle, O lies inside ABC and $\angle APC = \angle APO + \angle OPC = 60^\circ + 90^\circ = 150^\circ$, so by the Law of Cosines

$$s^2 = AC^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos 150^\circ = 25 + 12\sqrt{3}.$$

(If O lies outside the triangle, the triangle fits in a half-disk of radius 5, so its altitude is at most 5 and $s^2 \leq \frac{100}{3}$, which is smaller.)

The area is $\frac{\sqrt{3}}{4}\, s^2 = \frac{\sqrt{3}}{4}\left(25 + 12\sqrt{3}\right) = 9 + \frac{25}{4}\sqrt{3}$, so $a + b + c + d = 9 + 25 + 4 + 3 = 41$.

14. Complex numbers $a, b,$ and c are the zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to $a, b,$ and c in the complex plane are the vertices of a right triangle with hypotenuse h. Find h^2.

Solution:

Since $P(z)$ has no z^2 term, $a + b + c = 0$. Say the right angle is at b; then the hypotenuse joins a and c, so $h = |a - c|$, and $b = -(a + c)$. The midpoint $d = \frac{a+c}{2}$ of the hypotenuse is the circumcenter of the right triangle, so $|b - d| = \frac{h}{2}$. Since $b - d = -\frac{3}{2}(a + c)$, this gives $|a - c| = 3\,|a + c|$.

By the parallelogram law, $|a|^2 + |c|^2 = \frac{|a-c|^2 + |a+c|^2}{2}$, and $|b|^2 = |a + c|^2$, so

$$250 = \frac{9\,|a+c|^2 + |a+c|^2}{2} + |a+c|^2 = 6\,|a+c|^2.$$

Therefore $h^2 = |a - c|^2 = 9\,|a + c|^2 = \frac{9 \cdot 250}{6} = 375$.

15. There are n mathematicians seated around a circular table with n seats numbered $1, 2, 3, \ldots, n$ in clockwise order. After a break they again sit around the table. The mathematicians note that there is a positive integer a such that

(1) for each k, the mathematician who was seated in seat k before the break is seated in seat ka after the break (where seat $i + n$ is seat i);

(2) for every pair of mathematicians, the number of mathematicians sitting between them after the break, counting in both the clockwise and the counterclockwise directions, is different from either of the number of mathematicians sitting between them before the break.

Find the number of possible values of n with $1 < n < 1000$.

Solution:

Condition (1) requires the seats $a, 2a, \ldots, na$ to be pairwise distinct modulo n, which happens if and only if $\gcd(a, n) = 1$. For condition (2), the two mathematicians from seats i and j have gap counts before the break determined by $\pm(i - j) \bmod n$ and after the break by $\pm a(i - j) \bmod n$, so the requirement is $a(i - j) \not\equiv \pm(i - j) \pmod{n}$ for all $i \neq j$. Equivalently, $(a - 1)(i - j) \not\equiv 0$ and $(a + 1)(i - j) \not\equiv 0 \pmod{n}$ for every nonzero residue $i - j$, which holds exactly when $a - 1$ and $a + 1$ are also relatively prime to n.

So n is possible if and only if some a satisfies $\gcd\left((a - 1)\, a\, (a + 1),\ n\right) = 1$. Any three consecutive integers include a multiple of 2 and a multiple of 3, so no a works when $\gcd(n, 6) > 1$. Conversely, if $\gcd(n, 6) = 1$, then $a = 3$ works, since $2 \cdot 3 \cdot 4 = 24$ has only the prime factors 2 and 3.

The valid n with $1 < n < 1000$ are those congruent to $\pm 1 \pmod{6}$, namely $6k \pm 1$ for $1 \leq k \leq 166$, and there are $2 \cdot 166 = 332$ of them.

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