2010 AIME II Solutions

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1. Let N be the greatest integer multiple of 36 all of whose digits are even and no two of whose digits are the same. Find the remainder when N is divided by 1000.

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Solution:

Since $36 = 4 \cdot 9$, the number N must be divisible by both 4 and 9. Its digits are distinct members of $\{0, 2, 4, 6, 8\}$, whose total is 20, so N cannot use all five. The digit sum must be a multiple of 9, and being even it must be 18; the only such digit sets are $\{4, 6, 8\}$ and $\{0, 4, 6, 8\}$.

The largest number formed from $\{0, 4, 6, 8\}$ is 8640, which ends in 40, a multiple of 4. So $N = 8640$, and the remainder upon division by 1000 is 640.

2. A point P is chosen at random in the interior of a unit square S. Let $d(P)$ denote the distance from P to the closest side of S. The probability that $\frac{1}{5} \le d(P) \le \frac{1}{3}$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

The points with $d(P) \ge t$ form a concentric square of side $1 - 2t$. So $d(P) \ge \frac{1}{5}$ puts P inside the concentric square of side $\frac{3}{5}$, and $d(P) \le \frac{1}{3}$ keeps P outside the open concentric square of side $\frac{1}{3}$.

Since the unit square has area 1, the probability is the area between those two squares:

$$\left(\frac{3}{5}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{9}{25} - \frac{1}{9} = \frac{81 - 25}{225} = \frac{56}{225}.$$

Thus $m + n = 56 + 225 = 281.$

3. Let K be the product of all factors $(b - a)$ (not necessarily distinct) where a and b are integers satisfying $1 \le a < b \le 20$. Find the greatest positive integer n such that 2^n divides K.

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Solution:

For each value $v = b - a$, the pairs $(a, b) = (1, v + 1), (2, v + 2), \dots, (20 - v, 20)$ show that v occurs exactly $20 - v$ times, so $K = \prod_{v=1}^{19} v^{20-v}$. The exponent of 2 in K is therefore $\sum_v (20 - v)\, e(v)$, where $e(v)$ is the exponent of 2 in v.

Only even v contribute: $v = 2, 6, 10, 14, 18$ give $e = 1$; $v = 4, 12$ give $e = 2$; $v = 8$ gives $e = 3$; and $v = 16$ gives $e = 4$. The total is

$$18 + 16 \cdot 2 + 14 + 12 \cdot 3 + 10 + 8 \cdot 2 + 6 + 4 \cdot 4 + 2 = 150,$$

so $n = 150.$

4. Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

Number the gates 1 through 12. All $12 \cdot 11 = 132$ ordered pairs of distinct (old, new) gates are equally likely, and Dave walks 400 feet or less exactly when the gate numbers differ by at most 4.

A gate i has $\min(i+4, 12) - \max(i-4, 1)$ qualifying new gates: gates 1 and 12 have 4 each, gates 2 and 11 have 5, gates 3 and 10 have 6, gates 4 and 9 have 7, and gates 5 through 8 have 8 each. The total is $2(4+5+6+7) + 4 \cdot 8 = 76$.

The probability is $\frac{76}{132} = \frac{19}{33}$, so $m+n = 19+33 = 52$.

5. Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

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Solution:

Let $a = \log_{10} x, b = \log_{10} y$, and $c = \log_{10} z$. Taking logs of $xyz = 10^{81}$ gives $a+b+c = 81$. Since $\log_{10} yz = b+c$, the second condition is $a(b+c) + bc = ab + ac + bc = 468$.

Squaring the sum,

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+ac+bc) = 81^2 - 2 \cdot 468 = 6561 - 936 = 5625,$$

so the requested value is $\sqrt{5625} = 75$.

6. Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

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Solution:

If there is a linear factor, then some integer b is a root, so $b^4 - nb + 63 = 0$ and $n = b^3 + \frac{63}{b}$, forcing $b \mid 63$ and $b > 0$. The smallest value is 48, at $b = 3$.

Otherwise the polynomial splits into two quadratics, which we may take monic; since the x^3 coefficient vanishes, they have the form

$$(x^2 + px + q)(x^2 - px + r) = x^4 + (q + r - p^2)x^2 + p(r - q)x + qr.$$

Matching coefficients gives $q + r = p^2$, $qr = 63$, and $n = p(q - r)$. The factor pairs of 63 with square sum are $\{7, 9\}$ (sum 16, so $p = 4$) and $\{1, 63\}$ (sum 64, so $p = 8$), giving $n = 4 \cdot 2 = 8$ or $n = 8 \cdot 62 = 496$.

The smallest positive value overall is $n = 8$; indeed $(x^2 + 4x + 9)(x^2 - 4x + 7) = x^4 - 8x + 63$.

7. Let $P(z) = z^3 + az^2 + bz + c$, where $a, b,$ and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i, w + 9i,$ and $2w - 4,$ where $i^2 = -1.$ Find $|a + b + c|.$

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Solution:

Write $w = x + yi$ with x, y real. The sum of the roots is $4w + 12i - 4 = -a,$ which is real, so $4y + 12 = 0$ and $y = -3.$ The roots are then $x, x + 6i,$ and $2x - 4 - 6i.$ Because the coefficients are real, the two non-real roots must be conjugates, so $2x - 4 = x,$ giving $x = 4.$ The roots are $4, 4 + 6i,$ and $4 - 6i.$

Now

$$1 + a + b + c = P(1) = (1 - 4)\big(1 - (4 + 6i)\big)\big(1 - (4 - 6i)\big) = (-3)(9 + 36) = -135,$$

so $a + b + c = -136$ and $|a + b + c| = 136.$

8. Let N be the number of ordered pairs of nonempty sets $\mathcal{A}$ and $\mathcal{B}$ that have the following properties: • $\mathcal{A} \cup \mathcal{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, • $\mathcal{A} \cap \mathcal{B} = \emptyset$, • the number of elements of $\mathcal{A}$ is not an element of $\mathcal{A}$, • the number of elements of $\mathcal{B}$ is not an element of $\mathcal{B}$. Find N.

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Solution:

Let $k = |\mathcal{A}|$, so $|\mathcal{B}| = 12 - k$ with $1 \le k \le 11$. Since every element lies in exactly one set, $k \notin \mathcal{A}$ means $k \in \mathcal{B}$, and $12 - k \notin \mathcal{B}$ means $12 - k \in \mathcal{A}$. If $k = 6$, then 6 would have to belong to both sets, which is impossible, so $k \neq 6$.

For each other k, the elements k and $12 - k$ are already placed, and the remaining $k - 1$ elements of $\mathcal{A}$ can be chosen from the other 10 numbers in $\binom{10}{k-1}$ ways, with $\mathcal{B}$ taking the rest. Hence

$$N = \sum_{k=1}^{11} \binom{10}{k-1} - \binom{10}{5} = 2^{10} - 252 = 772.$$

9. Let $ABCDEF$ be a regular hexagon. Let G, H, I, J, K, and L be the midpoints of sides AB, BC, CD, DE, EF, and AF, respectively. The segments AH, BI, CJ, DK, EL, and FG bound a smaller regular hexagon. Let the ratio of the area of the smaller hexagon to the area of $ABCDEF$ be expressed as a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Center the hexagon at the origin with circumradius $1 : A = (1, 0), B = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), C = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $F = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Then $H = \left(0, \frac{\sqrt{3}}{2}\right)$ and $G = \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$. Rotation by 60° permutes the six segments, so the smaller hexagon is regular and concentric, and the area ratio is the square of the ratio of distances from the center to a vertex.

One vertex is the intersection of AH and FG. Line AH is $x + \frac{2}{\sqrt{3}}\, y = 1$, and line FG is $y = 3\sqrt{3}\, x - 2\sqrt{3}$. Substituting gives $x + 6x - 4 = 1$, so $x = \frac{5}{7}$ and $y = \frac{\sqrt{3}}{7}$.

That vertex has squared distance $\frac{25}{49} + \frac{3}{49} = \frac{4}{7}$ from the center, while A is at distance 1. The ratio of areas is $\frac{4}{7}$, and $m + n = 4 + 7 = 11$.

10. Find the number of second-degree polynomials $f(x)$ with integer coefficients and integer zeros for which $f(0) = 2010$.

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Solution:

Write $f(x) = a(x - r)(x - s)$ with integer roots r, s; such a polynomial is determined by a and the unordered pair $\{r, s\}$. The condition $f(0) = 2010$ says $a \cdot rs = 2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Since 2010 is squarefree, each of the four primes goes entirely to one of $|a|, |r|, |s|$.

First suppose $|r| \neq |s|$. Choosing which $k \geq 1$ of the four primes divide the roots ($\binom{4}{k}$ ways) and splitting those primes between the two roots (2^{k-1} unordered ways) gives $\sum_{k=1}^{4} \binom{4}{k} 2^{k-1} = 4 + 12 + 16 + 8 = 40$ choices of magnitudes. For each, the four sign patterns $(+, +), (+, -), (-, +), (-, -)$ of (r, s) are distinct and each forces the sign of a, giving $4 \cdot 40 = 160$ polynomials.

If $|r| = |s|$, squarefreeness forces $|r| = |s| = 1$, so $|a| = 2010$: the options are roots $1, 1$ or $-1, -1$ with $a = 2010$, or roots $1, -1$ with $a = -2010$, adding 3 more. In total $160 + 3 = 163$.

11. Define a *T-grid* to be a 3×3 matrix which satisfies the following two properties: (1) exactly five of the entries are 1's, and the remaining four entries are 0's, and (2) among the eight rows, columns, and long diagonals (the long diagonals are $\{a_{13}, a_{22}, a_{31}\}$ and $\{a_{11}, a_{22}, a_{33}\}$), no more than one of the eight has all three entries equal. Find the number of distinct T-grids.

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Solution:

There are $\binom{9}{5} = 126$ matrices satisfying (1); we subtract those with two or more constant lines. Two lines of 0's are impossible (they would need at least 5 zeros), and a line of 1's and a line of 0's cannot cross, so they must be parallel rows or parallel columns; likewise two lines of 1's cannot be parallel (6 ones), so they must cross, using exactly $3+3-1=5$ ones.

Case 1: a line of 1's and a parallel line of 0's. There are 6 choices for the all-1 row or column, 2 for the parallel all-0 line, and 3 ways to fill the remaining parallel line with two 1's and one $0:$ $6 \cdot 2 \cdot 3 = 36$ matrices. Every perpendicular line then contains both a 1 and a $0,$ so no third constant line appears and nothing is double-counted.

Case 2: two crossing lines of 1's and 0's elsewhere. The pair can be a row and a column ($3 \cdot 3 = 9$), a row or column with a diagonal ($6 \cdot 2 = 12$), or the two diagonals (1), for 22 matrices; one checks the four remaining 0's never form a constant line. So $126 - 36 - 22 = 68.$

12. Two noncongruent integer-sided isosceles triangles have the same perimeter and the same area. The ratio of the lengths of the bases of the two triangles is $8:7$. Find the minimum possible value of their common perimeter.

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Solution:

Since the integer bases are in ratio $8:7$, they are $8a$ and $7a$ for a positive integer a. Equal areas make the corresponding altitudes inversely proportional to the bases, say $7h$ and $8h$. The legs are then $\sqrt{16a^2+49h^2}$ and $\sqrt{\frac{49}{4}a^2+64h^2}$, and equal perimeters give

$$8a+2\sqrt{16a^2+49h^2}=7a+2\sqrt{\tfrac{49}{4}a^2+64h^2}.$$

Moving $7a$ to the left and squaring yields $a\sqrt{16a^2+49h^2}=15h^2-4a^2$; squaring again and simplifying leaves $225h^4=169a^2h^2$, so $h=\frac{13a}{15}$. The legs become

$$\sqrt{16a^2+49\cdot\tfrac{169a^2}{225}}=\frac{109a}{15}\qquad\text{and}\qquad\sqrt{\tfrac{49}{4}a^2+64\cdot\tfrac{169a^2}{225}}=\frac{233a}{30}.$$

For all sides to be integers, 30 must divide a. Taking $a=30$ gives the triangles $(218,218,240)$ and $(233,233,210)$, each with perimeter 676 and area 21840. The minimum common perimeter is 676.

13. The 52 cards in a deck are numbered $1, 2, \ldots, 52$. Alex, Blair, Corey, and Dylan each picks a card from the deck without replacement and with each card being equally likely to be picked. The two persons with lower numbered cards form a team, and the two persons with higher numbered cards form another team. Let $p(a)$ be the probability that Alex and Dylan are on the same team, given that Alex picks one of the cards a and $a+9$, and Dylan picks the other of these two cards. The minimum value of $p(a)$ for which $p(a) \geq \frac{1}{2}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

Condition on Alex and Dylan holding a and $a+9$. Blair and Corey then draw 2 of the remaining 50 cards, and Alex and Dylan are teammates exactly when both of those cards are below a (Alex and Dylan are the high team) or both are above $a+9$ (the low team). There are $a-1$ cards below and $52-(a+9)=43-a$ cards above, so

$$p(a) = \frac{\binom{a-1}{2} + \binom{43-a}{2}}{\binom{50}{2}}.$$

The numerator is $\frac{(a-1)(a-2)+(43-a)(42-a)}{2} = a^2 - 44a + 904$, so $p(a) \geq \frac{1}{2}$ becomes $a^2 - 44a + 904 \geq \frac{25 \cdot 49}{2}$, that is, $(a-22)^2 \geq \frac{385}{2}$. Since a is an integer, $|a-22| \geq 14$, so $a \leq 8$ or $a \geq 36$.

The parabola is smallest at the admissible points closest to $a = 22 : p(8) = p(36) = \frac{\binom{7}{2}+\binom{35}{2}}{\binom{50}{2}} = \frac{616}{1225} = \frac{88}{175}$, which is indeed at least $\frac{1}{2}$. Thus $m+n = 88+175 = 263$.

14. In right triangle ABC with the right angle at C, $\angle BAC < 45^\circ$ and $AB = 4$. Point P on $\overline{AB}$ has the properties that $\angle APC = 2\angle ACP$ and $CP = 1$. The ratio $\frac{AP}{BP}$ can be represented in the form $p + q\sqrt{r}$, where $p, q,$ and r are positive integers and r is not divisible by the square of any prime. Find $p + q + r$.

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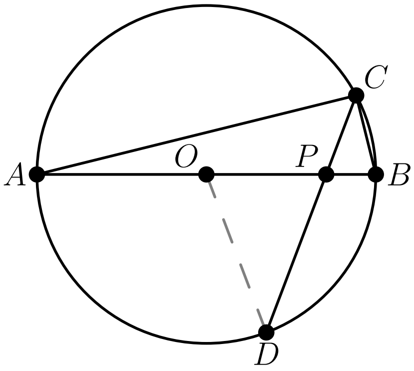
Solution:



Because the right angle is at C, segment AB is a diameter of the circumcircle; let O be its center, so the radius is 2. Let $\alpha = \angle ACP$ and extend $\overline{CP}$ to meet the circle again at D. The central angle over arc AD is $\angle AOD = 2\angle ACD = 2\alpha$, while vertical angles give $\angle DPB = \angle APC = 2\alpha$. So $\overline{OD}$ and $\overline{PD}$ make equal angles with line AB, and triangle ODP is isosceles with $DP = DO = 2$.

By the power of the point P,

$$AP \cdot PB = CP \cdot PD = 1 \cdot 2 = 2, \qquad AP + PB = 4,$$

so AP and PB are the roots of $t^2 - 4t + 2$, namely $2 \pm \sqrt{2}$. Since $\angle BAC < 45^\circ$, we have $BC < AC$ with $AC^2 + BC^2 = 16$, so $AC > 2\sqrt{2}$, and the triangle inequality gives $AP \geq AC - CP > 2\sqrt{2} - 1 > 2 - \sqrt{2}$. Hence $AP = 2 + \sqrt{2}$.

Therefore

$$\frac{AP}{BP} = \frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{(2+\sqrt{2})^2}{2} = 3 + 2\sqrt{2},$$

and $p + q + r = 3 + 2 + 2 = 7$.

15. In triangle ABC, $AC = 13$, $BC = 14$, and $AB = 15$. Points M and D lie on $\overline{AC}$ with $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on $\overline{AB}$ with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the other point of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets $\overline{BC}$ at Q. The ratio $\frac{BQ}{CQ}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m - n$.

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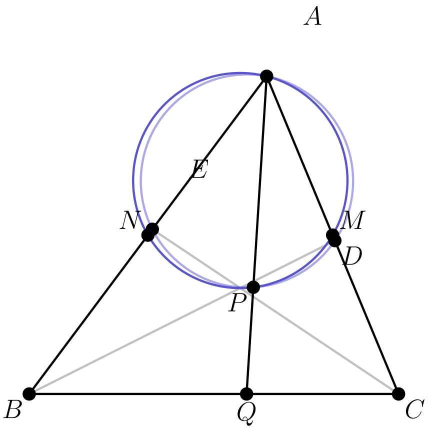
Solution:



By the angle bisector theorem, $AE = \frac{13}{27} \cdot 15$ and $CD = \frac{14}{29} \cdot 13$, so E lies on $\overline{AN}$ and D lies on $\overline{MC}$, with

$$NE = AN - AE = \frac{15}{2} - \frac{65}{9} = \frac{5}{18}, \qquad MD = CM - CD = \frac{13}{2} - \frac{182}{29} = \frac{13}{58}.$$

Since $AMPN$ is cyclic, $\angle ENP = \angle ANP = 180^\circ - \angle AMP = \angle DMP$, and since $AEPD$ is cyclic, $\angle NEP = 180^\circ - \angle AEP = \angle ADP = \angle MDP$. Hence triangles ENP and DMP are similar, so $\frac{NP}{MP} = \frac{NE}{MD}$. By the law of sines in triangles ANP and AMP, whose angles $\angle ANP$ and $\angle AMP$ are supplementary,

$$\frac{\sin \angle BAQ}{\sin \angle CAQ} = \frac{\sin \angle NAP}{\sin \angle MAP} = \frac{NP}{MP} = \frac{5/18}{13/58} = \frac{145}{117}.$$

Comparing the areas of triangles ABQ and ACQ, which share the cevian $\overline{AQ}$,

$$\frac{BQ}{CQ} = \frac{[ABQ]}{[ACQ]} = \frac{AB \sin \angle BAQ}{AC \sin \angle CAQ} = \frac{15}{13} \cdot \frac{145}{117} = \frac{725}{507},$$

which is in lowest terms since $507 = 3 \cdot 13^2$ and $725 = 5^2 \cdot 29$. Thus $m - n = 725 - 507 = 218$.

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