2010 AIME I Solutions

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1. Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

Since $2010^2 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 67^2$, it has $(2+1)^4 = 81$ positive divisors. A divisor is a perfect square exactly when each of its four exponents is 0 or 2, giving $2^4 = 16$ perfect squares and $81 - 16 = 65$ non-squares.

The probability of picking one of each is

$$p = \frac{16 \cdot 65}{\binom{81}{2}} = \frac{1040}{3240} = \frac{26}{81},$$

so $m + n = 26 + 81 = 107.$

2. Find the remainder when

$$9 \cdot 99 \cdot 999 \cdot \cdots \cdot \underbrace{99\ldots9}_{\text{999 9's}}$$

is divided by 1000.

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Solution:

Work modulo 1000. Every factor from the third one on ends in at least three 9s, so each is $\equiv -1 \pmod{1000}$. There are 999 factors in all, hence 997 of them are $\equiv -1$.

The product is therefore

$$\equiv 9 \cdot 99 \cdot (-1)^{997} \equiv -891 \equiv 109 \pmod{1000},$$

so the remainder is 109.

3. Suppose that $y = \frac{3}{4}x$ and $x^y = y^x$. The quantity $x + y$ can be expressed as a rational number $\frac{r}{s}$, where r and s are relatively prime positive integers. Find $r + s$.

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Solution:

Substituting $y = \frac{3}{4}x$ into $x^y = y^x$ gives

$$x^{\frac{3}{4}x} = \left(\tfrac{3}{4}x\right)^x.$$

Taking xth roots (the quantities here are positive), $x^{3/4} = \frac{3}{4}x$, so dividing by x yields $x^{-1/4} = \frac{3}{4}$, that is, $x = \left(\frac{4}{3}\right)^4 = \frac{256}{81}$.

Then $y = \frac{3}{4} \cdot \frac{256}{81} = \frac{64}{27}$, and

$$x + y = \frac{256}{81} + \frac{192}{81} = \frac{448}{81}.$$

Since $\gcd(448, 81) = 1$, the answer is $448 + 81 = 529$.

4. Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Let $p(h)$ be the probability that one player flips h heads. Splitting according to the two fair coins and the biased coin,

$$p(0) = \tfrac{1}{4} \cdot \tfrac{3}{7} = \tfrac{3}{28}, \quad p(1) = \tfrac{2}{4} \cdot \tfrac{3}{7} + \tfrac{1}{4} \cdot \tfrac{4}{7} = \tfrac{10}{28}, \quad p(2) = \tfrac{1}{4} \cdot \tfrac{3}{7} + \tfrac{2}{4} \cdot \tfrac{4}{7} = \tfrac{11}{28}, \quad p(3) = \tfrac{1}{4} \cdot \tfrac{4}{7} = \tfrac{4}{28}.$$

Jackie's and Phil's flips are independent with the same distribution, so the probability that their head counts agree is

$$\sum_h p(h)^2 = \frac{3^2 + 10^2 + 11^2 + 4^2}{28^2} = \frac{246}{784} = \frac{123}{392}.$$

Thus $m + n = 123 + 392 = 515$.

5. Positive integers a, b, c, and d satisfy $a > b > c > d$, $a + b + c + d = 2010$, and $a^2 - b^2 + c^2 - d^2 = 2010$. Find the number of possible values of a.

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Solution:

Factoring,

$$a^2 - b^2 + c^2 - d^2 = (a - b)(a + b) + (c - d)(c + d) \geq (a + b) + (c + d) = 2010,$$

since $a - b \geq 1$ and $c - d \geq 1$. Equality holds, so $a - b = c - d = 1$, that is, $b = a - 1$ and $d = c - 1$. Then $2010 = a + (a - 1) + c + (c - 1)$ gives $a + c = 1006$.

The condition $b > c$ means $a - 1 > c = 1006 - a$, so $a \geq 504$, and $d \geq 1$ means $c \geq 2$, so $a \leq 1004$. Every a in this range works, via $(a, b, c, d) = (a,\ a - 1,\ 1006 - a,\ 1005 - a)$.

The count is $1004 - 504 + 1 = 501$.

6. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying

$$x^2 - 2x + 2 \leq P(x) \leq 2x^2 - 4x + 3$$

for all real numbers x, and suppose $P(11) = 181$. Find $P(16)$.

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Solution:

Completing the square, the condition reads

$$(x-1)^2 + 1 \leq P(x) \leq 2(x-1)^2 + 1.$$

At $x = 1$ both bounds equal 1, so $P(1) = 1$. The quadratic $P(x) - \big((x-1)^2 + 1\big)$ is nonnegative for all x and vanishes at $x = 1$, so $x = 1$ is a double root: $P(x) = a(x-1)^2 + 1$ for some constant a.

From $P(11) = 100a + 1 = 181$ we get $a = \frac{9}{5}$. Then

$$P(16) = \frac{9}{5} \cdot 225 + 1 = 405 + 1 = 406.$$

7. Define an ordered triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of sets to be *minimally intersecting* if $|\mathcal{A} \cap \mathcal{B}| = |\mathcal{B} \cap \mathcal{C}| = |\mathcal{C} \cap \mathcal{A}| = 1$ and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = \emptyset$. For example, $(\{1,2\}, \{2,3\}, \{1,3,4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1,2,3,4,5,6,7\}$. Find the remainder when N is divided by 1000.

Note: $|\mathcal{S}|$ represents the number of elements in the set $\mathcal{S}$.

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Solution:

Write $\mathcal{A} \cap \mathcal{B} = \{x\}$, $\mathcal{B} \cap \mathcal{C} = \{y\}$, and $\mathcal{C} \cap \mathcal{A} = \{z\}$. Since $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = \emptyset$, the elements x, y, z are distinct, and they can be chosen in $7 \cdot 6 \cdot 5 = 210$ ways.

Each of the remaining 4 elements must not create any further pairwise intersections, so it can belong to exactly one of $\mathcal{A}, \mathcal{B}, \mathcal{C}$, or to none of them: 4 choices each, for $4^4 = 256$ assignments.

Hence $N = 210 \cdot 256 = 53760$, and the remainder upon division by 1000 is 760.

8. For a real number a, let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points (x, y) such that

$$\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25.$$

The region $\mathcal{R}$ is completely contained in a disk of radius r (a disk is the union of a circle and its interior). The minimum value of r can be written as $\frac{\sqrt{m}}{n}$, where m and n are integers and m is not divisible by the square of any prime. Find $m + n$.

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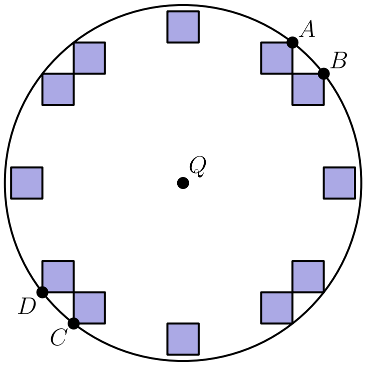
Solution:

Since $\lfloor x \rfloor$ and $\lfloor y \rfloor$ are integers whose squares sum to 25, the pair $(\lfloor x \rfloor, \lfloor y \rfloor)$ is one of the 12 pairs $(\pm 5, 0), (0, \pm 5), (\pm 3, \pm 4), (\pm 4, \pm 3)$. So $\mathcal{R}$ is the union of the 12 unit squares whose lower-left corners are these points.



The map $(x, y) \mapsto (1 - x, 1 - y)$ permutes these squares, so $\mathcal{R}$ is symmetric under 180° rotation about $Q = \left(\frac{1}{2}, \frac{1}{2}\right)$. The smallest enclosing disk is unique, so its center must be Q. The farthest points of $\mathcal{R}$ from Q are square corners such as $A = (4, 5)$ and $B = (5, 4)$, at distance

$$\sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{7}{2}\right)^2} = \frac{\sqrt{130}}{2};$$

checking all twelve squares confirms no corner is farther.

Hence the minimum radius is $r = \frac{\sqrt{130}}{2}$, and $m + n = 130 + 2 = 132$.

9. Let (a, b, c) be a real solution of the system of equations

$$x^3 - xyz = 2, \qquad y^3 - xyz = 6, \qquad z^3 - xyz = 20.$$

The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Adding xyz to each equation gives $x^3 = 2 + xyz, y^3 = 6 + xyz$, and $z^3 = 20 + xyz$. Let $P = xyz$. Multiplying the three equations yields

$$P^3 = (2 + P)(6 + P)(20 + P) = P^3 + 28P^2 + 172P + 240,$$

so $28P^2 + 172P + 240 = 0$, i.e. $7P^2 + 43P + 60 = 0$, whose roots are $P = -\frac{15}{7}$ and $P = -4$. Each root is achievable: the cube roots of $2 + P, 6 + P, 20 + P$ then really do have product P.

Adding the original equations, $x^3 + y^3 + z^3 = 28 + 3P$, which is maximized by the larger root $P = -\frac{15}{7}$:

$$a^3 + b^3 + c^3 = 28 - \frac{45}{7} = \frac{151}{7}.$$

Thus $m + n = 151 + 7 = 158.$

10. Let N be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the a_i's are integers, and $0 \le a_i \le 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

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Solution:

Write each coefficient as $a_i = 10b_i + c_i$ with digits $b_i, c_i \in \{0, 1, \ldots, 9\}$; every integer $0 \le a_i \le 99$ splits this way uniquely. Setting $m = b_3b_2b_1b_0$ and $n = c_3c_2c_1c_0$ (read as base-10 numbers), the condition becomes

$$2010 = 10m + n.$$

Conversely, any nonnegative integers m, n with $10m + n = 2010$ satisfy $m \le 201$ and $n \le 2010$, so each has at most four digits; those digits recover the b_i and c_i, hence the a_i. So representations correspond exactly to choices of $m \in \{0, 1, \ldots, 201\}$ with $n = 2010 - 10m$, and $N = 202$.

11. Let $\mathcal{R}$ be the region consisting of the set of points in the coordinate plane that satisfy both $|8 - x| + y \le 10$ and $3y - x \ge 15$. When $\mathcal{R}$ is revolved around the line whose equation is $3y - x = 15$, the volume of the resulting solid is $\frac{m\pi}{n\sqrt{p}}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.

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Solution:

The condition $|8 - x| + y \le 10$ means $y \le x + 2$ for $x \le 8$ and $y \le 18 - x$ for $x \ge 8$. Intersecting with the half-plane $3y - x \ge 15$ leaves the triangle with vertices $A = \left(\frac{9}{2}, \frac{13}{2}\right)$ and $B = \left(\frac{39}{4}, \frac{33}{4}\right)$ on the line $3y - x = 15$, and apex $C = (8, 10)$.

Side AB lies on the axis of revolution, and the foot D of the perpendicular from C to the line, namely $(8.7, 7.9)$, lies between A and B. So the solid is two cones sharing a base of radius CD with heights summing to AB, and its volume is $\frac{1}{3}\pi \cdot CD^2 \cdot AB$. Here

$$CD = \frac{|3 \cdot 10 - 8 - 15|}{\sqrt{10}} = \frac{7}{\sqrt{10}}, \qquad AB = \sqrt{\left(\tfrac{21}{4}\right)^2 + \left(\tfrac{7}{4}\right)^2} = \frac{7\sqrt{10}}{4}.$$

The volume is $\frac{1}{3}\pi \cdot \frac{49}{10} \cdot \frac{7\sqrt{10}}{4} = \frac{343\pi}{12\sqrt{10}}$, so $m + n + p = 343 + 12 + 10 = 365$.

12. Let $m \geq 3$ be an integer and let $S = \{3, 4, 5, \ldots, m\}$. Find the smallest value of m such that for every partition of S into two subsets, at least one of the subsets contains integers a, b, and c (not necessarily distinct) such that $ab = c$.

Note: a partition of S is a pair of sets A, B such that $A \cap B = \emptyset$ and $A \cup B = S$.

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Solution:

First, $m = 243$ works. Suppose $S = \{3, 4, \ldots, 243\}$ were partitioned into T and U with neither containing a product, and say $3 \in T$. Then $9 = 3 \cdot 3$ must lie in U, so $81 = 9 \cdot 9$ must lie in T, and then $243 = 3 \cdot 81$ must lie in U. Now consider 27 : if $27 \in T$, then $3 \cdot 27 = 81$ puts a product in T; if $27 \in U$, then $9 \cdot 27 = 243$ puts one in U. Either way we reach a contradiction.

For $m = 242$, the partition $T = \{3, \ldots, 8\} \cup \{81, \ldots, 242\}$ and $U = \{9, \ldots, 80\}$ avoids products: two elements of $\{3, \ldots, 8\}$ multiply to something in $[9, 64] \subseteq U$, any product involving an element of $\{81, \ldots, 242\}$ is at least $3 \cdot 81 = 243 > 242$, and two elements of U multiply to at least $81 > 80$.

Hence the smallest such m is 243.

13. Rectangle $ABCD$ and a semicircle with diameter $\overline{AB}$ are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment $\overline{AB}$, and segment $\overline{CD}$ at distinct points N, U, and T, respectively. Line ℓ divides region $\mathcal{R}$ into two regions with areas in the ratio $1 : 2$. Suppose that $AU = 84, AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

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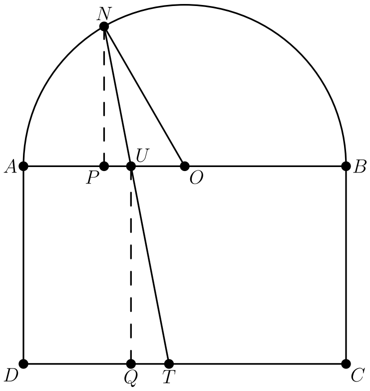
Solution:

Here $AB = 84 + 168 = 252$, so the semicircle has center O (the midpoint of $\overline{AB}$) and radius 126. Since $AN = AO = ON = 126$, triangle AON is equilateral, so $\angle AON = 60^\circ$ and sector AON is exactly one third of the semicircle. Likewise, if Q is the foot of the perpendicular from U to $\overline{DC}$, then $AU : UB = 1 : 2$ makes rectangle $AUQD$ one third of rectangle $ABCD$.



The part of $\mathcal{R}$ on the A-side of ℓ equals (sector AON) $- [NUO] +$ (rectangle $AUQD$) $+ [UQT]$. Since this must be one third of $\mathcal{R}$, we need $[NUO] = [UQT]$. Let P be the foot of the perpendicular from N to $\overline{AB}$. In the 30-60-90 triangle NOP, $OP = 63$ and $NP = 63\sqrt{3}$, so $UP = 84 - 63 = 21$ and $UO = 126 - 84 = 42$. Triangles NUP and TUQ are similar right triangles (vertical angles at U), so

$$\frac{[UQT]}{[NUP]} = \left(\frac{UQ}{NP}\right)^2, \qquad \frac{[NUO]}{[NUP]} = \frac{UO}{UP} = 2,$$

the latter because both triangles have height NP over bases UO and UP.

Setting the two triangle areas equal gives $\left(\frac{UQ}{NP}\right)^2 = 2$, so

$$DA = UQ = NP\sqrt{2} = 63\sqrt{3} \cdot \sqrt{2} = 63\sqrt{6},$$

and $m + n = 63 + 6 = 69$.

14. For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \leq 300$.

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

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Solution:

Each term $\lfloor \log_{10}(kn) \rfloor$ is nondecreasing in n, so f is nondecreasing and we just locate where it passes 300. For $n = 100:$ the products kn run from 100 to 10^4, giving $\lfloor \log_{10} \rfloor = 2$ for $k \leq 9, 3$ for $10 \leq k \leq 99$, and 4 for $k = 100$, so $f(100) = 9 \cdot 2 + 90 \cdot 3 + 4 = 292$.

For $n = 109:$ since $9 \cdot 109 = 981 < 1000$ and $91 \cdot 109 = 9919 < 10^4$, the terms are 2 for $k \leq 9, 3$ for $10 \leq k \leq 91$, and 4 for $92 \leq k \leq 100:$

$$f(109) = 9 \cdot 2 + 82 \cdot 3 + 9 \cdot 4 = 300.$$

For $n = 110:$ now $91 \cdot 110 = 10010 \geq 10^4$, so ten terms equal 4 and $f(110) = 18 + 81 \cdot 3 + 10 \cdot 4 = 301 > 300$.

By monotonicity, the largest valid n is 109.

15. In $\triangle ABC$ with $AB = 12, BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$.

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Solution:

Let $k = \frac{AM}{CM}$. Triangles ABM and CBM share the altitude from B, so $\frac{[ABM]}{[CBM]} = k$. Since the inradius of a triangle is its area divided by its semiperimeter, equal inradii force $\frac{12+AM+BM}{13+CM+BM} = k$ as well. From $AM + CM = 15$ we get $AM = \frac{15k}{k+1}$ and $CM = \frac{15}{k+1}$; since $AM = k \cdot CM$, the perimeter equation simplifies to $BM(1 - k) = 13k - 12$, so

$$BM = \frac{13k - 12}{1 - k},$$

and $BM > 0$ forces $\frac{12}{13} < k < 1$.

Stewart's theorem on cevian $\overline{BM}$ gives $AB^2 \cdot CM + BC^2 \cdot AM = AC\left(BM^2 + AM \cdot CM\right)$, so

$$BM^2 = \frac{144 + 169k}{k + 1} - \frac{225k}{(k + 1)^2}.$$

Setting this equal to $\frac{(13k-12)^2}{(1-k)^2}$ and clearing denominators yields $(169k^2 + 88k + 144)(1 - k)^2 = (13k - 12)^2(k + 1)^2$, which simplifies to $4k\left(69k^2 - 112k + 44\right) = 0$.

The roots are $k = 0, k = \frac{2}{3}$, and $k = \frac{22}{23}$, and only $k = \frac{22}{23}$ exceeds $\frac{12}{13}$ (then $AM = \frac{22}{3}, CM = \frac{23}{3}$, $BM = 10$). Hence $p + q = 22 + 23 = 45$.

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