2009 AIME II Solutions

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1. Before starting to paint, Bill had 130 ounces of blue paint, 164 ounces of red paint, and 188 ounces of white paint. Bill painted four equally sized stripes on a wall, making a blue stripe, a red stripe, a white stripe, and a pink stripe. Pink is a mixture of red and white, not necessarily in equal amounts. When Bill finished, he had equal amounts of blue, red, and white paint left. Find the total number of ounces of paint Bill had left.

Solution:

Say each stripe used s ounces of paint. Blue was used only on the blue stripe, so s ounces of blue were used. Since the three leftovers are equal and the colors started 34 and 58 ounces apart, red use exceeded blue use by $164 - 130 = 34$ ounces and white use exceeded blue use by $188 - 130 = 58$ ounces. That extra red and white is exactly the pink stripe, so $s = 34 + 58 = 92.$

Bill therefore had $130 - 92 = 38$ ounces of each color left, for a total of $3 \cdot 38 = 114$ ounces.

2. Suppose that $a, b,$ and c are positive real numbers such that $a^{\log_3 7} = 27, b^{\log_7 11} = 49,$ and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

Solution:

By the power rule for exponents,

$$a^{(\log_3 7)^2} = \left(a^{\log_3 7}\right)^{\log_3 7} = 27^{\log_3 7} = \left(3^{\log_3 7}\right)^3 = 7^3 = 343.$$

In the same way, $b^{(\log_7 11)^2} = 49^{\log_7 11} = \left(7^{\log_7 11}\right)^2 = 11^2 = 121$, and $c^{(\log_{11} 25)^2} = \left(\sqrt{11}\right)^{\log_{11} 25} = \left(11^{\log_{11} 25}\right)^{1/2} = 25^{1/2} = 5.$

The sum is $343 + 121 + 5 = 469.$

3. In rectangle $ABCD$, $AB = 100$. Let E be the midpoint of $\overline{AD}$. Given that line AC and line BE are perpendicular, find the greatest integer less than AD.

Solution:

Let $AD = h$, and place $A = (0,0), B = (100,0), C = (100,h), D = (0,h)$, so $E = \left(0, \frac{h}{2}\right)$. Line AC has slope $\frac{h}{100}$ and line BE has slope $\frac{h/2}{-100} = -\frac{h}{200}$. Perpendicularity gives

$$\frac{h}{100} \cdot \left(-\frac{h}{200}\right) = -1,$$

so $h^2 = 20000$ and $h = 100\sqrt{2} \approx 141.42.$

The greatest integer less than AD is $141.$

4. A group of children held a grape-eating contest. When the contest was over, the winner had eaten n grapes, and the child in kth place had eaten $n+2-2k$ grapes. The total number of grapes eaten in the contest was 2009. Find the smallest possible value of n.

Solution:

Let c be the number of children. The grape counts $n, n-2, \ldots, n+2-2c$ form an arithmetic sequence, so the total is c times the average of the first and last terms:

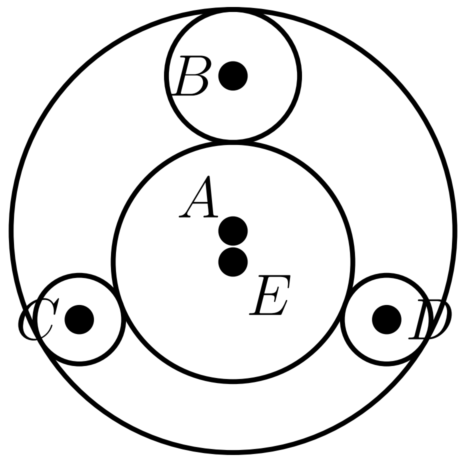
$$c \cdot \frac{n+(n+2-2c)}{2} = c(n+1-c) = 2009 = 7^2 \cdot 41.$$

Thus $c \mid 2009$ and $n = \frac{2009}{c} + c - 1$.

The last-place child ate $n+2-2c = \frac{2009}{c} + 1 - c \geq 0$ grapes, which forces $c(c-1) \leq 2009$, ruling out $c = 49, 287,$ and 2009. The remaining divisors give $n = 2009$ for $c = 1, n = 287 + 6 = 293$ for $c = 7$, and $n = 49 + 40 = 89$ for $c = 41$.

The smallest possible value is $n = 89$.

5. Equilateral triangle T is inscribed in circle A, which has radius 10. Circle B with radius 3 is internally tangent to circle A at one vertex of T. Circles C and D, both with radius 2, are internally tangent to circle A at the other two vertices of T. Circles B, C, and D are all externally tangent to circle E, which has radius $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution:

Place the center of circle A at the origin with the triangle's vertices at $(0,10)$ and $\left(\pm 5\sqrt{3},-5\right)$. A circle internally tangent to A at a vertex has its center on the radius to that vertex, so circle B has center $(0,7)$ and circles C and D have centers $\left(\mp 4\sqrt{3},-4\right)$ (at distance $10-2=8$ from the origin).

By symmetry the center of circle E, of radius r, lies on the y-axis at $(0,y)$. External tangency to B gives $7-y=r+3$, so $y=4-r$. External tangency to C gives

$$\left(4\sqrt{3}\right)^2+(4-r+4)^2=(r+2)^2,$$

that is, $48+(8-r)^2=(r+2)^2$, which simplifies to $112-16r=4r+4$, so $r=\frac{27}{5}$.

Then $m+n=27+5=32$.

6. Let m be the number of five-element subsets that can be chosen from the set of the first 14 natural numbers so that at least two of the five numbers are consecutive. Find the remainder when m is divided by 1000.

Solution:

Count the complement: subsets $a_1 < a_2 < a_3 < a_4 < a_5$ with no two consecutive. Setting $b_i = a_i - (i-1)$ turns each such subset into five distinct numbers $b_1 < b_2 < \cdots < b_5$ in $\{1, \ldots, 10\}$, and this map is reversible, so there are $\binom{10}{5} = 252$ subsets with no two consecutive numbers.

Therefore $m = \binom{14}{5} - \binom{10}{5} = 2002 - 252 = 1750$, and the remainder upon division by 1000 is 750.

7. Define $n!!$ to be $n(n-2)(n-4)\cdots 3\cdot 1$ for n odd and $n(n-2)(n-4)\cdots 4\cdot 2$ for n even. When $\sum_{i=1}^{2009}\frac{(2i-1)!!}{(2i)!!}$ is expressed as a fraction in lowest terms, its denominator is $2^a b$ with b odd. Find $\frac{ab}{10}$.

Solution:

The ith term is $\frac{(2i-1)!!}{(2i)!!}$ with odd numerator, and $(2i)!! = 2^i \cdot i!$. Because $\binom{2i}{i} = \frac{(2i)!}{i!\,i!} = \frac{2^i(2i-1)!!}{i!}$ is an integer, every odd prime power dividing $i!$ also divides $(2i-1)!!$. Hence in lowest terms the ith term has denominator exactly 2^{a_i} where $a_i = i + e_i$ and e_i is the exponent of 2 in $i!$. The a_i strictly increase, so over the common denominator $2^{a_{2009}}$ every term except the last contributes an even numerator while the last contributes an odd one. The sum in lowest terms therefore has denominator exactly $2^{a_{2009}}$, so $b = 1$.

By Legendre's formula,

$$e_{2009} = 1004 + 502 + 251 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 2001,$$

so $a = 2009 + 2001 = 4010$. Then $\frac{ab}{10} = \frac{4010\cdot 1}{10} = 401$.

8. Dave rolls a fair six-sided die until a six appears for the first time. Independently, Linda rolls a fair six-sided die until a six appears for the first time. Let m and n be relatively prime positive integers such that $\frac{m}{n}$ is the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die. Find $m+n$.

Solution:

The probability that a player's first six appears on roll k is $p_k = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$. The probability of a tie is

$$\sum_{k=1}^{\infty} p_k^2 = \frac{1}{36} \cdot \frac{1}{1-\frac{25}{36}} = \frac{1}{11}.$$

The probability that Linda needs exactly one more roll than Dave is $\sum_{k=1}^{\infty} p_k p_{k+1} = \frac{5}{6} \sum_{k=1}^{\infty} p_k^2 = \frac{5}{66}$, and by symmetry the same holds with the players swapped. The total probability is $\frac{1}{11} + 2 \cdot \frac{5}{66} = \frac{6+10}{66} = \frac{8}{33}$, so $m+n = 8+33 = 41$.

9. Let m be the number of solutions in positive integers to the equation $4x + 3y + 2z = 2009$, and let n be the number of solutions in positive integers to the equation $4x + 3y + 2z = 2000$. Find the remainder when $m - n$ is divided by 1000.

Solution:

If (x, y, z) is a positive solution of $4x + 3y + 2z = 2009$, then $(x-1, y-1, z-1)$ is a nonnegative solution of $4x + 3y + 2z = 2000$, and conversely, since $4 + 3 + 2 = 9$. So m equals the number of nonnegative solutions of $4x + 3y + 2z = 2000$, and $m - n$ counts the nonnegative solutions of that equation in which at least one variable is 0.

If $x = 0 : 3y + 2z = 2000$ forces y even, $0 \le y \le 666$, giving 334 solutions. If $y = 0 : 2x + z = 1000$ with $0 \le x \le 500$ gives 501. If $z = 0 : 4x + 3y = 2000$ forces $y \equiv 0 \pmod 4$, $0 \le y \le 664$, giving 167. The solutions $(0, 0, 1000)$ and $(500, 0, 0)$ are each counted twice, so

$$m - n = 334 + 501 + 167 - 2 = 1000.$$

The remainder upon division by 1000 is 0.

10. Four lighthouses are located at points A, B, C, and D. The lighthouse at A is 5 kilometers from the lighthouse at B, the lighthouse at B is 12 kilometers from the lighthouse at C, and the lighthouse at A is 13 kilometers from the lighthouse at C. To an observer at A, the angle determined by the lights at B and D and the angle determined by the lights at C and D are equal. To an observer at C, the angle determined by the lights at A and B and the angle determined by the lights at D and B are equal. The number of kilometers from A to D is given by $\frac{p\sqrt{r}}{q}$, where p, q, and r are relatively prime positive integers, and r is not divisible by the square of any prime. Find $p + q + r$.

Solution:

Since $5^2 + 12^2 = 13^2$, angle B is right. Place $A = (0, 0), B = (5, 0), C = (5, 12)$. The condition at A says $\angle BAD = \angle CAD$, so D lies on the bisector of angle BAC. Using the half-angle formula with $\tan \angle BAC = \frac{12}{5}$,

$$\tan \frac{\angle BAC}{2} = \frac{\sin \angle BAC}{1 + \cos \angle BAC} = \frac{12/13}{1 + 5/13} = \frac{2}{3},$$

so D lies on the line $y = \frac{2}{3}x$.

The condition at C says CB bisects angle ACD, so ray CD is the reflection of ray CA over line CB, which is the vertical line $x = 5$. The reflection of A is $(10, 0)$, so D lies on the line through $C = (5, 12)$ and $(10, 0)$, namely $5y = 120 - 12x$.

Solving $y = \frac{2}{3}x$ and $5y = 120 - 12x$ gives $x = \frac{180}{23}, y = \frac{120}{23}$. Then

$$AD = \frac{60}{23}\sqrt{3^2 + 2^2} = \frac{60\sqrt{13}}{23},$$

so $p + q + r = 60 + 23 + 13 = 96$.

11. For certain pairs (m, n) of positive integers with $m \geq n$ there are exactly 50 distinct positive integers k such that $|\log m - \log k| < \log n$. Find the sum of all possible values of the product mn.

Solution:

The inequality $|\log m - \log k| < \log n$ is equivalent to $\frac{m}{n} < k < mn$. Write $m = nq + r$ with $0 \leq r < n$; since $m \geq n \geq 2$ (for $n = 1$ no k works), $q \geq 1$. The integers k in the interval are $q+1, q+2, \ldots, mn-1$, so there are $mn - q - 1 = 50$ of them, that is, $mn - q = 51$, or

$$q(n^2 - 1) + nr = 51.$$

For $n \geq 8$ the left side is at least 63, so $2 \leq n \leq 7$. Checking each case, only $n = 2$, $r = 0, q = 17$ (so $m = 34$) and $n = 3, r = 1, q = 6$ (so $m = 19$) work. These give $mn = 68$ and $mn = 57$; indeed $17 < k < 68$ and $\frac{19}{3} < k < 57$ each contain exactly 50 integers.

The sum of all possible values of mn is $68 + 57 = 125.$

12. From the set of integers $\{1, 2, 3, \ldots, 2009\}$, choose k pairs $\{a_i, b_i\}$ with $a_i < b_i$ so that no two pairs have a common element. Suppose that all the sums $a_i + b_i$ are distinct and less than or equal to 2009. Find the maximum possible value of k.

Solution:

Let $S = \sum_{i=1}^{k}(a_i + b_i)$. The $2k$ chosen elements are distinct positive integers, so $S \geq 1 + 2 + \cdots + 2k = k(2k+1)$. The k sums are distinct integers at most 2009, so $S \leq 2009 + 2008 + \cdots + (2010 - k) = \frac{k(4019-k)}{2}$. Combining,

$$k(2k+1) \leq \frac{k(4019-k)}{2} \implies 4k + 2 \leq 4019 - k \implies k \leq \frac{4017}{5} = 803.4,$$

so $k \leq 803$.

To achieve $k = 803$, take the pairs $(i,\ 1206 + i)$ for $1 \leq i \leq 401$, whose sums are the even numbers $1208, 1210, \ldots, 2008$, together with the pairs $(a,\ a + 403)$ for $402 \leq a \leq 803$, whose sums are the odd numbers $1207, 1209, \ldots, 2009$. The elements used are 1–803, 805–1607 with no repeats, and all 803 sums are distinct and at most 2009.

The maximum is $k = 803$.

13. Let A and B be the endpoints of a semicircular arc of radius 2. The arc is divided into seven congruent arcs by six equally spaced points $C_1, C_2, \ldots, C_6$. All chords of the form $\overline{AC_i}$ or $\overline{BC_i}$ are drawn. Let n be the product of the lengths of these twelve chords. Find the remainder when n is divided by 1000.

Solution:

Put the circle in the complex plane with center 0, $A = -2, B = 2$, and $C_i = 2\omega^i$ for $i = 1, \ldots, 6$, where $\omega = e^{i\pi/7}$. Then $AC_i = 2\left|\omega^i + 1\right|$ and $BC_i = 2\left|\omega^i - 1\right|$, so

$$AC_i \cdot BC_i = 4\left|\omega^{2i} - 1\right|.$$

As i runs over $1, \ldots, 6$, the numbers $\omega^{2i} = e^{2\pi i \cdot i/7}$ run over all six nontrivial 7th roots of unity ζ^j. Since $\prod_{j=1}^{6}(x - \zeta^j) = 1 + x + \cdots + x^6$, plugging in $x = 1$ gives $\prod_{j=1}^{6}\left|1 - \zeta^j\right| = 7$. Therefore

$$n = \prod_{i=1}^{6} 4\left|\omega^{2i} - 1\right| = 4^6 \cdot 7 = 28672.$$

The remainder when n is divided by 1000 is 672.

14. The sequence (a_n) satisfies $a_0 = 0$ and $a_{n+1} = \frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - a_n^2}$ for $n \geq 0$. Find the greatest integer less than or equal to a_{10}.

Solution:

Write $a_n = 2^n \sin\theta_n$ with $\theta_0 = 0$, so $\sqrt{4^n - a_n^2} = 2^n\,|\cos\theta_n|$. Let $\theta = \arcsin\frac{3}{5}$, so $\cos\theta = \frac{4}{5}$. The recursion becomes

$$a_{n+1} = 2^{n+1}\left(\cos\theta\sin\theta_n + \sin\theta\,|\cos\theta_n|\right) = 2^{n+1}\sin(\theta_n \pm \theta),$$

with the plus sign when $\cos\theta_n \geq 0$ and the minus sign when $\cos\theta_n < 0$.

Since $\frac{1}{2} < \frac{3}{5} < \frac{\sqrt{2}}{2}$, we have $30^\circ < \theta < 45^\circ$. The angles $\theta, 2\theta$ have positive cosine, so the sequence of angles runs $0, \theta, 2\theta, 3\theta$. But $90^\circ < 3\theta < 135^\circ$ has negative cosine, so $\theta_4 = 2\theta$, and from then on the angle alternates between 3θ and 2θ. In particular $\theta_n = 2\theta$ for every even $n \geq 2$.

With $\sin 2\theta = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$,

$$a_{10} = 2^{10}\sin 2\theta = 1024 \cdot \frac{24}{25} = \frac{24576}{25} = 983.04,$$

so the answer is 983.

15. Let $\overline{MN}$ be a diameter of a circle with diameter 1. Let A and B be points on one of the semicircular arcs determined by $\overline{MN}$ such that A is the midpoint of the semicircle and $MB = \frac{3}{5}$. Point C lies on the other semicircular arc. Let d be the length of the line segment whose endpoints are the intersections of diameter $\overline{MN}$ with the chords $\overline{AC}$ and $\overline{BC}$. The largest possible value of d can be written in the form $r - s\sqrt{t}$, where $r, s,$ and t are positive integers and t is not divisible by the square of any prime. Find $r + s + t$.

Solution:

Let chords BC and AC meet $\overline{MN}$ at P and Q, and set $x = \frac{CM}{CN}$. Since $\angle MBN = 90^\circ$ (angle in a semicircle) and $MB = \frac{3}{5}$, we get $BN = \frac{4}{5}$; also $AM = AN = \frac{\sqrt{2}}{2}$. Because P lies on both MN and BC, the ratio $\frac{MP}{PN}$ equals the ratio of the distances from M and N to line BC, i.e. $\frac{[BMC]}{[BNC]}$. In cyclic quadrilateral $MBNC$ the angles BMC and BNC are supplementary, so their sines are equal and

$$\frac{MP}{PN} = \frac{BM \cdot MC}{BN \cdot NC} = \frac{3x}{4}, \qquad \frac{MQ}{QN} = \frac{AM \cdot MC}{AN \cdot NC} = x.$$

Since $MN = 1$, these give $MP = \frac{3x}{3x+4}$ and $MQ = \frac{x}{x+1}$, so

$$d = MQ - MP = \frac{x}{x+1} - \frac{3x}{3x+4} = \frac{x}{3x^2 + 7x + 4} = \frac{1}{3x + \frac{4}{x} + 7}.$$

As C ranges over the far semicircle, x takes every positive value. By AM-GM, $3x + \frac{4}{x} \geq 2\sqrt{12} = 4\sqrt{3}$, with equality at $x = \frac{2}{\sqrt{3}}$. Hence the largest value of d is

$$\frac{1}{7 + 4\sqrt{3}} = 7 - 4\sqrt{3},$$

since $(7 + 4\sqrt{3})(7 - 4\sqrt{3}) = 1$. Then $r + s + t = 7 + 4 + 3 = 14$.

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