2009 AIME I Solutions

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1. Call a 3-digit number *geometric* if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

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Solution:

Write the digits as a, ar, ar^2. For the largest geometric number, take $a = 9$. An integer ratio at least 2 would push the next digit past 9, and $r = 1$ repeats digits, so r is a fraction whose denominator squares into 9 : the choices $r = \frac{2}{3}$ and $r = \frac{1}{3}$ give 964 and 931. The largest is 964.

For the smallest, take hundreds digit 1. Then the tens digit r must be an integer at least 2 (the digits are distinct), and $r = 2$ gives 124, which beats $r = 3$'s 139.

The difference is $964 - 124 = 840$.

2. There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.

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Solution:

Write $z = a + 164i$. Clearing the denominator gives $z = 4i(z+n)$, that is,

$$a + 164i = 4i\,(a + n + 164i) = -656 + 4(a+n)i.$$

Real parts give $a = -656$, and imaginary parts give $164 = 4(a+n)$, so $a + n = 41$ and $n = 41 + 656 = 697$.

3. A coin that comes up heads with probability $p > 0$ and tails with probability $1 - p > 0$ independently on each flip is flipped eight times. Suppose the probability of three heads and five tails is equal to $\frac{1}{25}$ of the probability of five heads and three tails. Let $p = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

The condition says

$$\binom{8}{3} p^3 (1-p)^5 = \frac{1}{25} \binom{8}{5} p^5 (1-p)^3.$$

Since $\binom{8}{3} = \binom{8}{5}$ and both p and $1 - p$ are positive, dividing by $p^3(1-p)^3$ leaves $(1-p)^2 = \frac{p^2}{25}$, so $1 - p = \frac{p}{5}$. Hence $p = \frac{5}{6}$, and $m + n = 5 + 6 = 11$.

4. In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$, and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.

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Solution:

Place A at the origin and let $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{d} = \overrightarrow{AD}$, so that $C = \mathbf{b} + \mathbf{d}$, $M = \frac{17}{1000}\mathbf{b}$, and $N = \frac{17}{2009}\mathbf{d}$. Since P lies on $\overline{AC}$, write $P = s\,(\mathbf{b} + \mathbf{d})$ where $s = \frac{AP}{AC}$; since P also lies on line MN, write $P = tM + (1-t)N$ for some t.

Because $\mathbf{b}$ and $\mathbf{d}$ are independent, the coefficients must agree:

$$s = \frac{17t}{1000} \qquad \text{and} \qquad s = \frac{17(1-t)}{2009}.$$

Thus $t = \frac{1000s}{17}$ and $1 - t = \frac{2009s}{17}$; adding gives $1 = \frac{3009s}{17}$.

Therefore $\frac{AC}{AP} = \frac{1}{s} = \frac{3009}{17} = 177.$

5. Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

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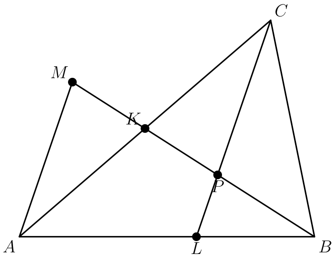
Solution:



Because $AK = CK$ and K is the midpoint of $\overline{PM}$, the diagonals of quadrilateral $APCM$ bisect each other, so $APCM$ is a parallelogram and $AM \parallel CP$. Since P lies on line CL and B, P, M all lie on line BK, triangles BLP and BAM are similar.

Thus $\frac{LP}{AM} = \frac{BL}{BA}$. The angle bisector theorem gives $\frac{AL}{LB} = \frac{AC}{BC} = \frac{450}{300} = \frac{3}{2}$, so $\frac{BL}{BA} = \frac{2}{2+3} = \frac{2}{5}$.

Therefore $LP = \frac{2}{5} \cdot AM = \frac{2}{5} \cdot 180 = 72.$

6. How many positive integers N less than 1000 are there such that the equation $x^{\lfloor x \rfloor} = N$ has a solution for x? (The notation $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)

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Solution:

Suppose $\lfloor x \rfloor = k$ for a positive integer k. As x runs over $[k, k+1)$, the value x^k increases continuously from k^k toward $(k+1)^k$, so the attainable integers N are exactly those with $k^k \leq N \leq (k+1)^k - 1$: there are $(k+1)^k - k^k$ of them, and these ranges are disjoint for different k. (Values of x below 1 produce no new positive integers, since $x^0 = 1$ is already attained.)

For $k = 1, 2, 3, 4$ the counts are $2 - 1 = 1, 9 - 4 = 5, 64 - 27 = 37$, and $625 - 256 = 369$, and every such N is at most $624 < 1000$. For $k = 5$ the smallest value is $5^5 = 3125 > 1000$.

The total is $1 + 5 + 37 + 369 = 412.$

7. The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)} - 1 = \frac{1}{n+\frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

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Solution:

The relation says $5^{a_{n+1}-a_n} = 1 + \frac{3}{3n+2} = \frac{3n+5}{3n+2}$. Multiplying these equations for $n = 1, 2, \ldots, k-1$ telescopes:

$$5^{a_k - a_1} = \frac{3k+2}{5}, \qquad \text{so} \qquad a_k = 1 + \log_5 \frac{3k+2}{5} = \log_5(3k+2).$$

Thus a_k is an integer exactly when $3k+2$ is a power of 5. Since $5^j \equiv 2^j \pmod 3$, only odd exponents j give numbers of the form $3k+2$. The power $5^1 = 5$ gives $k = 1$, which is excluded, and the next, $5^3 = 125 = 3 \cdot 41 + 2$, gives $k = 41$.

8. Let $S = \{2^0, 2^1, 2^2, \ldots, 2^{10}\}$. Consider all possible positive differences of pairs of elements of S. Let N be the sum of all of these differences. Find the remainder when N is divided by 1000.

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Solution:

In the sum of all positive differences, the element 2^k is added once for each smaller element (k times) and subtracted once for each larger element ($10 - k$ times). Hence

$$N = \sum_{k=0}^{10} (2k - 10)\, 2^k = 2 \sum_{k=0}^{10} k\, 2^k - 10 \sum_{k=0}^{10} 2^k.$$

The standard sums are $\sum_{k=0}^{10} k\, 2^k = 9 \cdot 2^{11} + 2 = 18434$ and $\sum_{k=0}^{10} 2^k = 2^{11} - 1 = 2047$, so $N = 2 \cdot 18434 - 10 \cdot 2047 = 36868 - 20470 = 16398$.

The remainder upon division by 1000 is 398.

9. A game show offers a contestant three prizes A, B and C, each of which is worth a whole number of dollars from $\$1$ to $\$9999$ inclusive. The contestant wins the prizes by correctly guessing the price of each prize in the order A, B, C. As a hint, the digits of the three prices are given. On a particular day, the digits given were $1, 1, 1, 1, 3, 3, 3$. Find the total number of possible guesses for all three prizes consistent with the hint.

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Solution:

Concatenating the three guessed prices in order produces an arrangement of the seven given digits, and each guess is recovered uniquely from an arrangement together with a way to cut it into three consecutive nonempty blocks of at most four digits each (prices run from $\$1$ to $\$9999$, and no price can start with 0 here since every digit is 1 or 3). There are $\frac{7!}{4!\,3!} = 35$ arrangements of four 1s and three 3s.

The ordered block lengths are the ways to write 7 as an ordered sum of three parts between 1 and 4 : the permutations of $(1, 2, 4)$, $(2, 2, 3)$, and $(1, 3, 3)$, giving $6 + 3 + 3 = 12$ cuts for each arrangement.

The total is $35 \cdot 12 = 420.$

10. The Annual Interplanetary Mathematics Examination (AIME) is written by a committee of five Martians, five Venusians, and five Earthlings. At meetings, committee members sit at a round table with chairs numbered from 1 to 15 in clockwise order. Committee rules state that a Martian must occupy chair 1 and an Earthling must occupy chair $15.$ Furthermore, no Earthling can sit immediately to the left of a Martian, no Martian can sit immediately to the left of a Venusian, and no Venusian can sit immediately to the left of an Earthling. The number of possible seating arrangements for the committee is $N \cdot (5!)^3$. Find N.

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Solution:

First choose which planet sits in each chair; the individuals from each planet can then be assigned to their chairs in $5!$ ways apiece, so N counts the planet patterns. The adjacency rules say exactly that, reading clockwise, each maximal block of Martians must be followed by a block of Venusians and then a block of Earthlings before Martians can appear again. Since chair 1 holds a Martian and chair 15 holds an Earthling, the chairs from 1 to 15 consist of the pattern (Martian block, Venusian block, Earthling block) repeated k times, for some $1 \leq k \leq 5.$

For a given k, each planet's five members are distributed into k nonempty blocks in order, and the number of ways to write 5 as an ordered sum of k positive integers is $\binom{4}{k-1}$. The three planets' block sizes are independent, so

$$N = \sum_{k=1}^{5} \binom{4}{k-1}^3 = 1^3 + 4^3 + 6^3 + 4^3 + 1^3 = 346.$$

11. Consider the set of all triangles OPQ where O is the origin and P and Q are distinct points in the plane with nonnegative integer coordinates (x, y) such that $41x + y = 2009$. Find the number of such distinct triangles whose area is a positive integer.

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Solution:

The points on the line with nonnegative integer coordinates are $P_i = (i,\ 2009 - 41i)$ for $i = 0, 1, \ldots, 49$ – fifty points in all. For $P = P_i$ and $Q = P_j$, the shoelace formula gives

$$[OPQ] = \frac{1}{2}\,|\,i(2009 - 41j) - j(2009 - 41i)\,| = \frac{2009}{2}\,|i - j|.$$

This is automatically positive for distinct points, and since 2009 is odd, it is an integer exactly when $i - j$ is even, that is, when i and j have the same parity. There are 25 even and 25 odd indices, so the number of triangles is $\binom{25}{2} + \binom{25}{2} = 300 + 300 = 600.$

12. In right $\triangle ABC$ with hypotenuse $\overline{AB}$, $AC = 12$, $BC = 35$, and $\overline{CD}$ is the altitude to $\overline{AB}$. Let ω be the circle having $\overline{CD}$ as a diameter. Let I be a point outside $\triangle ABC$ such that $\overline{AI}$ and $\overline{BI}$ are both tangent to circle ω. The ratio of the perimeter of $\triangle ABI$ to the length AB can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Because $\overline{CD} \perp \overline{AB}$ and D is an endpoint of the diameter, AB is tangent to ω at D. Together with the tangent lines AI and BI, this makes ω the inscribed circle of triangle ABI. Write $AD = y$, $BD = z$, and let x be the tangent length from I. The right-triangle altitude satisfies $CD^2 = AD \cdot BD$, so the inradius of ABI is $r = \frac{1}{2}\sqrt{yz}$.

With semiperimeter $s = x + y + z$, the tangent lengths are exactly $s - AB = x$, $s - BI = y$, and $s - AI = z$, so the area of ABI equals both rs and, by Heron's formula, $\sqrt{s \cdot xyz}$. Equating and squaring,

$$\frac{s^2\,yz}{4} = s\,xyz, \qquad \text{so} \qquad s = 4x,$$

which gives $AB = y + z = s - x = 3x$.

The perimeter is $2s = 8x$, so its ratio to AB is $\frac{8x}{3x} = \frac{8}{3}$ (independent of the given legs), and $m + n = 8 + 3 = 11$.

13. The terms of the sequence (a_i) defined by $a_{n+2} = \frac{a_n+2009}{1+a_{n+1}}$ for $n \geq 1$ are positive integers. Find the minimum possible value of $a_1 + a_2$.

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Solution:

Clearing denominators, $a_{n+2}(1 + a_{n+1}) = a_n + 2009$ for all $n \geq 1$. Subtracting each instance from the next gives

$$a_{n+2} - a_n = (a_{n+2} + 1)(a_{n+3} - a_{n+1}).$$

If some difference $a_{n+2} - a_n$ were nonzero, then every later difference would be nonzero as well, and since each $a_{n+2} + 1 \geq 2$, the identity would force $|a_3 - a_1| > |a_4 - a_2| > |a_5 - a_3| > \cdots$, an infinite strictly decreasing sequence of positive integers — impossible. Hence $a_{n+2} = a_n$ for all $n :$ the odd-indexed terms are all equal and the even-indexed terms are all equal, and any such choice of positive integers works.

The recursion then reads $a_1(1 + a_2) = a_1 + 2009$, so $a_1 a_2 = 2009 = 7^2 \cdot 41$. Among the factor pairs of 2009, the sum is smallest for $41 \cdot 49$, giving $41 + 49 = 90$.

14. For $t = 1, 2, 3, 4$, define $S_t = \sum_{i=1}^{350} a_i^t$, where $a_i \in \{1, 2, 3, 4\}$. If $S_1 = 513$ and $S_4 = 4745$, find the minimum possible value for S_2.

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Solution:

For $j = 1, 2, 3, 4$, let m_j be the number of a_i equal to j. Then

$$m_1 + m_2 + m_3 + m_4 = 350, \quad m_1 + 2m_2 + 3m_3 + 4m_4 = 513, \quad m_1 + 16m_2 + 81m_3 + 256m_4 = 4745.$$

Subtracting the first equation from the other two gives $m_2 + 2m_3 + 3m_4 = 163$ and $15m_2 + 80m_3 + 255m_4 = 4395$; subtracting 15 times the former from the latter leaves $50m_3 + 210m_4 = 1950$, that is, $5m_3 + 21m_4 = 195$. Hence m_4 is a nonnegative multiple of 5, and only $m_4 = 0$ and $m_4 = 5$ keep everything nonnegative, giving $(m_1, m_2, m_3, m_4) = (226, 85, 39, 0)$ or $(215, 112, 18, 5)$.

These yield $S_2 = m_1 + 4m_2 + 9m_3 + 16m_4 = 917$ and 905 respectively, so the minimum is 905.

15. In triangle ABC, $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. The maximum possible area of $\triangle BPC$ can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

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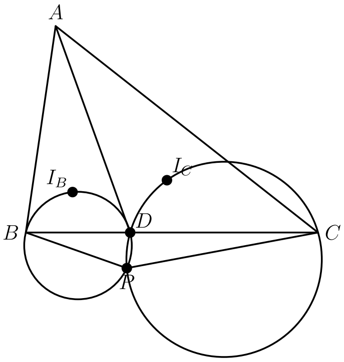
Solution:



In triangle ABD the incenter satisfies $\angle BI_BD = 90^\circ + \frac{\angle BAD}{2}$, and likewise $\angle CI_CD = 90^\circ + \frac{\angle DAC}{2}$, so these two angles sum to $180^\circ + \frac{\angle BAC}{2}$. The law of cosines gives $\cos \angle BAC = \frac{10^2+16^2-14^2}{2\cdot 10\cdot 16} = \frac{1}{2}$, so $\angle BAC = 60^\circ$ and the sum is 210°.

The second intersection point P lies on the opposite side of $\overline{BC}$ from the incenters (were it on the same side, the two cyclic quadrilaterals would force $\angle BPC = 210^\circ > 180^\circ$). Then BI_BDP and CI_CDP are convex cyclic quadrilaterals, so

$$\angle BPC = \angle BPD + \angle DPC = (180^\circ - \angle BI_BD) + (180^\circ - \angle CI_CD) = 360^\circ - 210^\circ = 150^\circ,$$

independent of D. Hence P moves along a fixed circular arc through B and C.

The area of triangle BPC is maximized at the midpoint of the arc, where $BP = PC = x$. The law of cosines gives $14^2 = 2x^2 + \sqrt{3}\,x^2$, so $x^2 = \frac{196}{2+\sqrt{3}} = 196\left(2-\sqrt{3}\right)$, and the area is $\frac{1}{2}x^2 \sin 150^\circ = 49\left(2-\sqrt{3}\right) = 98 - 49\sqrt{3}$. Thus $a + b + c = 98 + 49 + 3 = 150$.

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