2008 AIME II Solutions

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1. Let $N = 100^2 + 99^2 - 98^2 - 97^2 + 96^2 + \cdots + 4^2 + 3^2 - 2^2 - 1^2$, where the additions and subtractions alternate in pairs. Find the remainder when N is divided by 1000.

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Solution:

Group the terms four at a time. For $k = 1, 2, \ldots, 25$, the block ending at $(4k)^2$ is

$$(4k)^2 + (4k-1)^2 - (4k-2)^2 - (4k-3)^2 = 2(8k-2) + 2(8k-4) = 32k - 12,$$

using the difference of squares $a^2 - b^2 = (a+b)(a-b)$ with $a - b = 2$ twice.

Summing over $k = 1$ to 25,

$$N = 32 \cdot \frac{25 \cdot 26}{2} - 12 \cdot 25 = 10400 - 300 = 10100,$$

so the remainder when N is divided by 1000 is 100.

2. Rudolph bikes at a constant rate and stops for a five-minute break at the end of every mile. Jennifer bikes at a constant rate which is three-quarters the rate that Rudolph bikes, but Jennifer takes a five-minute break at the end of every two miles. Jennifer and Rudolph begin biking at the same time and arrive at the 50-mile mark at exactly the same time. How many minutes has it taken them?

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Solution:

Let Rudolph bike at r miles per minute. He rests after each of miles 1 through 49, so his total time is $\frac{50}{r} + 49 \cdot 5 = \frac{50}{r} + 245$ minutes. Jennifer bikes at $\frac{3r}{4}$ miles per minute and rests after each of miles $2, 4, \ldots, 48$, so her total time is $\frac{50}{3r/4} + 24 \cdot 5 = \frac{200}{3r} + 120$ minutes.

Setting the times equal gives

$$\frac{50}{r} + 245 = \frac{200}{3r} + 120, \qquad \text{so} \qquad 125 = \frac{200 - 150}{3r} = \frac{50}{3r},$$

and $r = \frac{2}{15}$. The common time is $\frac{50}{2/15} + 245 = 375 + 245 = 620$ minutes.

3. A block of cheese in the shape of a rectangular solid measures 10 cm by 13 cm by 14 cm. Ten slices are cut from the cheese. Each slice has a width of 1 cm and is cut parallel to one face of the cheese. The individual slices are not necessarily parallel to each other. What is the maximum possible volume in cubic cm of the remaining block of cheese after ten slices have been cut off?

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Solution:

Every slice is 1 cm wide and parallel to a face, so after each cut the remaining cheese is still a rectangular block, with one dimension shortened by $1.$ If the ten slices shorten the three dimensions by $p, q,$ and r with $p + q + r = 10,$ the remaining block measures $(10 - p) \times (13 - q) \times (14 - r),$ and these dimensions sum to $37 - 10 = 27.$

By the AM-GM inequality, a product of positive numbers with fixed sum 27 is greatest when all three are equal to $9,$ which is achieved by taking 1 slice from the 10 cm dimension, 4 from the 13 cm dimension, and 5 from the 14 cm dimension. The maximum volume is $9^3 = 729$ cubic cm.

4. There exist r unique nonnegative integers $n_1 > n_2 > \cdots > n_r$ and r unique integers a_k $(1 \leq k \leq r)$ with each a_k either 1 or -1 such that

$$a_1 3^{n_1} + a_2 3^{n_2} + \cdots + a_r 3^{n_r} = 2008.$$

Find $n_1 + n_2 + \cdots + n_r$.

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Solution:

In base $3, 2008 = 2202101_3$, that is,

$$2008 = 2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^3 + 3^2 + 3^0.$$

To convert the digits 2 into coefficients ± 1, use $2 \cdot 3^k = 3^{k+1} - 3^k$. The two adjacent digits 2 collapse neatly: $2 \cdot 3^6 + 2 \cdot 3^5 = (3^7 - 3^6) + (3^6 - 3^5) = 3^7 - 3^5$, and $2 \cdot 3^3 = 3^4 - 3^3$.

Therefore

$$2008 = 3^7 - 3^5 + 3^4 - 3^3 + 3^2 + 3^0,$$

which has distinct exponents and coefficients ± 1, as required. The sum of the exponents is $7 + 5 + 4 + 3 + 2 + 0 = 21$.

5. In trapezoid $ABCD$ with $\overline{BC} \parallel \overline{AD}$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37^\circ$, $\angle D = 53^\circ$, and M and N be the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively. Find the length MN.

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Solution:

Extend legs $\overline{AB}$ and $\overline{DC}$ until they meet at a point E. Since $\angle A + \angle D = 37^\circ + 53^\circ = 90^\circ$, triangle EAD has a right angle at E. Because $\overline{BC} \parallel \overline{AD}$, triangle EBC is the image of triangle EAD under a homothety centered at E, so the midpoint M of $\overline{BC}$ maps to the midpoint N of $\overline{AD}$; in particular E, M, and N are collinear.

The median to the hypotenuse of a right triangle is half the hypotenuse, so $EN = \frac{2008}{2} = 1004$ and $EM = \frac{1000}{2} = 500$. Therefore

$$MN = EN - EM = 1004 - 500 = 504.$$

6. The sequence $\{a_n\}$ is defined by

$$a_0 = 1, \quad a_1 = 1, \quad \text{and} \quad a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}} \quad \text{for } n \geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1, \quad b_1 = 3, \quad \text{and} \quad b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}} \quad \text{for } n \geq 2.$$

Find $\frac{b_{32}}{a_{32}}$.

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Solution:

Dividing the recurrence by a_{n-1} gives

$$\frac{a_n}{a_{n-1}} = 1 + \frac{a_{n-1}}{a_{n-2}},$$

so the consecutive-term ratio increases by exactly 1 each step. For $\{a_n\}$ the first ratio is $\frac{a_1}{a_0} = 1$, so $\frac{a_n}{a_{n-1}} = n$ and $a_n = n!$. The same computation applies to $\{b_n\}$, whose first ratio is $\frac{b_1}{b_0} = 3$, so $\frac{b_n}{b_{n-1}} = n + 2$ and $b_n = \frac{(n+2)!}{2}$.

Therefore

$$\frac{b_{32}}{a_{32}} = \frac{34!/2}{32!} = \frac{34 \cdot 33}{2} = 561.$$

7. Let $r, s,$ and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

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Solution:

The cubic has no x^2 term, so $r+s+t=0$ by Vieta's formulas. Hence $r+s=-t$, $s+t=-r$, and $t+r=-s$, and the desired sum is

$$(-t)^3 + (-r)^3 + (-s)^3 = -(r^3+s^3+t^3).$$

Whenever $r+s+t=0$, the identity $r^3+s^3+t^3-3rst = (r+s+t)(r^2+s^2+t^2-rs-st-tr)$ gives $r^3+s^3+t^3=3rst$. By Vieta's formulas, $rst = -\frac{2008}{8} = -251$, so $r^3+s^3+t^3=-753$, and the answer is $-(-753)=753$.

8. Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2a)\sin(na)]$$

is an integer.

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Solution:

By the product-to-sum identity, $2\cos(k^2a)\sin(ka) = \sin(k^2a + ka) - \sin(k^2a - ka) = \sin(k(k+1)a) - \sin((k-1)ka)$. Summing over $k = 1$ to n, the terms telescope, leaving

$$\sin(n(n+1)a) = \sin\frac{n(n+1)\pi}{2008}.$$

A sine is an integer only when it is $-1, 0,$ or $1,$ that is, when its argument is a multiple of $\frac{\pi}{2}$. So we need $\frac{n(n+1)}{2008}$ to be a multiple of $\frac{1}{2}$, i.e. $1004 \mid n(n+1)$, where $1004 = 4 \cdot 251$ and 251 is prime.

Since n and $n+1$ are coprime, 251 must divide one of them, so $n \geq 250$. For $n = 250$ the product $250 \cdot 251$ is not divisible by 4. For $n = 251$ the product $251 \cdot 252$ is divisible by $4 \cdot 251 = 1004$. The smallest such n is 251.

9. A particle is located on the coordinate plane at $(5, 0)$. Define a *move* for the particle as a counterclockwise rotation of $\pi/4$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p, q), find the greatest integer less than or equal to $|p| + |q|$.

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Solution:

Identify the plane with the complex plane, so a move sends z to $\omega z + 10$ with $\omega = e^{i\pi/4}$. Starting from $z_0 = 5$ and iterating,

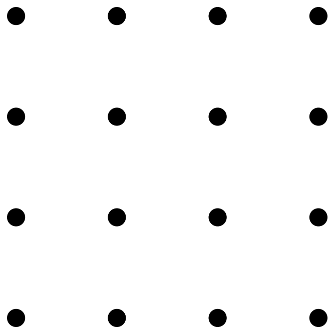
$$z_{150} = 5\omega^{150} + 10\left(\omega^{149} + \omega^{148} + \cdots + \omega + 1\right).$$

Since $\omega^8 = 1$ and $150 = 8 \cdot 18 + 6$, we get $\omega^{150} = \omega^6 = -i$. In the geometric sum, every block of 8 consecutive powers adds to 0, so the 150 terms reduce to $1 + \omega + \cdots + \omega^5 = -\omega^6 - \omega^7 = i - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$. Therefore

$$z_{150} = -5i + 10\left(-\frac{\sqrt{2}}{2} + \left(1 + \frac{\sqrt{2}}{2}\right)i\right) = -5\sqrt{2} + \left(5 + 5\sqrt{2}\right)i.$$

Thus $|p| + |q| = 5\sqrt{2} + 5 + 5\sqrt{2} = 5 + 10\sqrt{2} \approx 19.14$, and the greatest integer less than or equal to this is 19.

10. The diagram below shows a 4×4 rectangular array of points, each of which is 1 unit away from its nearest neighbors.



Define a *growing path* to be a sequence of distinct points of the array with the property that the distance between consecutive points of the sequence is strictly increasing. Let m be the maximum possible number of points in a growing path, and let r be the number of growing paths consisting of exactly m points. Find mr.

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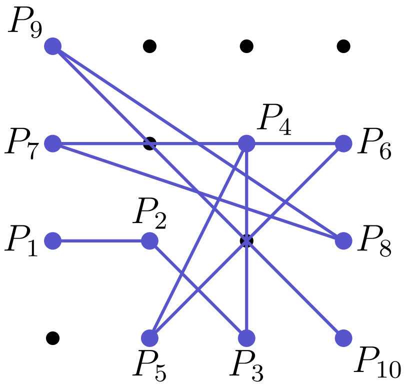
Solution:

The squared distance between two points of the array is $a^2 + b^2$, where a and b are the coordinate differences, each in $\{0, 1, 2, 3\}$ and not both zero. The possible values are $1, 2, 4, 5, 8, 9, 10, 13, 18$ — only 9 values — so a growing path has at most 10 points, and a path with 10 points must use all nine distances in increasing order. Label its points $P_1, \ldots, P_{10}$ so that $P_1P_2 = 1$ and $P_9P_{10} = \sqrt{18}$.

Since $\sqrt{18}$ is realized only by opposite corners, there are 4 ordered choices of (P_{10}, P_9). Next, $P_8P_9 = \sqrt{13}$ leaves 2 choices for P_8, the two neighbors of P_{10}, symmetric across the main diagonal. From there the distances $\sqrt{10}, 3, \sqrt{8}, \sqrt{5}, 2, \sqrt{2}$ force $P_7, P_6, \ldots, P_2$ uniquely (for P_7 the alternative corner choice fails because the point needed next for P_6 would coincide with P_9 or P_{10}). Finally P_1 must be at distance 1 from P_2, and 3 of its neighbors are unused. One of the resulting paths is shown below.



Hence $m = 10$ and $r = 4 \cdot 2 \cdot 3 = 24$, so $mr = 240$.

11. In triangle ABC, $AB = AC = 100$, and $BC = 56$. Circle P has radius 16 and is tangent to $\overline{AC}$ and $\overline{BC}$. Circle Q is externally tangent to circle P and is tangent to $\overline{AB}$ and $\overline{BC}$. No point of circle Q lies outside of $\triangle ABC$. The radius of circle Q can be expressed in the form $m - n\sqrt{k}$, where $m, n,$ and k are positive integers and k is the product of distinct primes. Find $m + nk$.

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Solution:

Place $B = (0,0)$ and $C = (56,0)$; the altitude from A has length $\sqrt{100^2 - 28^2} = 96$, so $A = (28,96)$. Then $\sin B = \frac{24}{25}$, $\cos B = \frac{7}{25}$, and

$$\tan\frac{B}{2} = \frac{\sin B}{1+\cos B} = \frac{3}{4} = \tan\frac{C}{2}.$$

A circle of radius r tangent to $\overline{BC}$ and to a slanted side has its center on the bisector from that base vertex, at height r and horizontal distance $\frac{r}{\tan(C/2)} = \frac{4r}{3}$ from the vertex. Thus $P = \left(56 - \frac{64}{3},\, 16\right)$ and $Q = \left(\frac{4q}{3},\, q\right)$, where q is the radius of circle Q.

External tangency means $PQ = q + 16:$

$$\left(\frac{104-4q}{3}\right)^2 + (16-q)^2 = (16+q)^2.$$

Since $(16+q)^2 - (16-q)^2 = 64q$, this becomes $(104-4q)^2 = 576q$, i.e. $(26 - q)^2 = 36q$, which simplifies to $q^2 - 88q + 676 = 0$, so $q = 44 \pm 6\sqrt{35}$.

The root $44 + 6\sqrt{35} \approx 79.5$ would make circle Q extend outside the triangle, so $q = 44 - 6\sqrt{35}$. Here $m = 44, n = 6$, and $k = 35 = 5 \cdot 7$, giving $m + nk = 44 + 210 = 254$.

12. There are two distinguishable flagpoles, and there are 19 flags, of which 10 are identical blue flags, and 9 are identical green flags. Let N be the number of distinguishable arrangements using all of the flags in which each flagpole has at least one flag and no two green flags on either pole are adjacent. Find the remainder when N is divided by 1000.

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Solution:

Suppose the first pole gets b blue and g green flags, the second the remaining $10-b$ blue and $9-g$ green. On a pole with b blue flags, the green flags must occupy distinct gaps among the $b+1$ gaps around the blues, in $\binom{b+1}{g}$ ways. Temporarily ignoring the requirement that each pole be nonempty, the total is

$$\sum_{b=0}^{10}\sum_{g=0}^{9}\binom{b+1}{g}\binom{11-b}{9-g}=\sum_{b=0}^{10}\binom{12}{9}=11\cdot 220=2420,$$

where the inner sum collapses by Vandermonde's identity, since $(b+1)+(11-b)=12$.

The arrangements that leave a pole empty put all 19 flags on one pole, in $\binom{11}{9}=55$ ways for each choice of pole. Hence

$$N=2420-2\cdot 55=2310,$$

and the remainder when N is divided by 1000 is 310.

13. A regular hexagon with center at the origin in the complex plane has opposite pairs of sides one unit apart. One pair of sides is parallel to the imaginary axis. Let R be the region outside the hexagon, and let $S = \left\{\frac{1}{z} \mid z \in R\right\}$. Then the area of S has the form $a\pi + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

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Solution:

The hexagon's sides lie at distance $\frac{1}{2}$ from the origin, with one side on the line $\operatorname{Re} z = \frac{1}{2}$, so R is the union of the six half-planes obtained by rotating $\operatorname{Re} z > \frac{1}{2}$ by multiples of 60°. If $w = u + vi = \frac{1}{z}$, then $\operatorname{Re} z = \operatorname{Re} \frac{1}{w} = \frac{u}{u^2+v^2} > \frac{1}{2}$ is equivalent to $u^2 + v^2 < 2u$, i.e. $(u-1)^2 + v^2 < 1$. So each half-plane maps onto an open unit disk, and S is the union of six unit disks centered at the sixth roots of unity.

Cut the plane into six 60° wedges by the rays at angles $30^\circ + 60^\circ k$; by symmetry, within each wedge S coincides with the disk whose center lies in that wedge. The rays at $\pm 30^\circ$ meet the circle $|w-1| = 1$ at $\left(\frac{3}{2}, \pm\frac{\sqrt{3}}{2}\right)$, so the piece of S in that wedge consists of two triangles with vertices at 0, the center 1, and one of these points — each isosceles with two sides 1 and apex angle 120°, area $\frac{\sqrt{3}}{4}$ — together with the 120° sector of the disk between them, area $\frac{\pi}{3}$.

Each wedge therefore contributes $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$, and the total area is

$$6\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = 2\pi + 3\sqrt{3} = 2\pi + \sqrt{27}.$$

Thus $a = 2, b = 27$, and $a + b = 29$.

14. Let a and b be positive real numbers with $a \geq b$. Let ρ be the maximum possible value of $\frac{a}{b}$ for which the system of equations

$$a^2 + y^2 = b^2 + x^2 = (a - x)^2 + (b - y)^2$$

has a solution (x, y) satisfying $0 \leq x < a$ and $0 \leq y < b$. Then ρ^2 can be expressed as a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Draw the rectangle with vertices $A = (0, 0), B = (a, 0), C = (a, b), D = (0, b)$, and let $E = (x, 0)$ on $\overline{AB}$ and $F = (a, b - y)$ on $\overline{BC}$. Then $DE^2 = b^2 + x^2, DF^2 = a^2 + y^2$, and $EF^2 = (a - x)^2 + (b - y)^2$, so the system says exactly that triangle DEF is equilateral, with the constraints keeping E and F on those two sides.

Let $\theta = \angle ADE$, so $x = b \tan \theta$ and $DE = \frac{b}{\cos \theta}$. Since $\angle EDF = 60^\circ$ and the corner angle at D is 90°, we get $\angle CDF = 30^\circ - \theta$, so $y = a \tan(30^\circ - \theta)$ and $DF = \frac{a}{\cos(30^\circ - \theta)}$. Setting $DE = DF$ gives

$$\frac{a}{b} = \frac{\cos(30^\circ - \theta)}{\cos \theta} = \cos 30^\circ + \sin 30^\circ \tan \theta,$$

which is increasing in θ. The requirement $y \geq 0$ forces $\theta \leq 30^\circ$.

The maximum is therefore at $\theta = 30^\circ$, where $\frac{a}{b} = \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$, attained with $y = 0$ and $x = \frac{b}{\sqrt{3}} < a$. Hence $\rho^2 = \frac{4}{3}$, and $m + n = 4 + 3 = 7$.

15. Find the largest integer n satisfying the following conditions: (i) n^2 can be expressed as the difference of two consecutive cubes; (ii) $2n+79$ is a perfect square.

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Solution:

Condition (i) says $n^2=(m+1)^3-m^3=3m^2+3m+1$ for some integer m. Multiplying by 4 and rearranging, $4n^2-1=12m^2+12m+3$, i.e.

$$(2n-1)(2n+1)=3(2m+1)^2.$$

The factors on the left are consecutive odd numbers, hence coprime, so one of them is a perfect square and the other is 3 times a square. If $2n-1=3k^2$, then $2n+1\equiv 2 \pmod 3$ would be a perfect square, which is impossible. Hence $2n-1=k^2$ with k odd.

Writing $k=2a+1$ gives $n=2a^2+2a+1$. Condition (ii) says $2n+79=4a^2+4a+81=d^2$, so

$$(d-2a-1)(d+2a+1)=d^2-(2a+1)^2=80.$$

The two factors have the same parity, so both are even: the pairs $(2,40)$, $(4,20)$, $(8,10)$ give $2a+1=19,8,1$, of which the odd values yield $a=9$ (so $n=181$) and $a=0$ (so $n=1$).

For $n=181:$ indeed $181^2=32761=105^3-104^3$ (here $2n+1=363=3\cdot 11^2$, as required), and $2n+79=441=21^2$. So the largest such n is 181.

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