2008 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Of the students attending a school party, 60% of the students are girls, and 40% of the students like to dance. After these students are joined by 20 more boy students, all of whom like to dance, the party is now 58% girls. How many students now at the party like to dance?

Solution:

Let x be the number of students originally at the party, so $0.6x$ are girls and $0.4x$ like to dance. When the 20 boys arrive, the number of girls is unchanged but the total becomes $x+20$, so $0.6x = 0.58(x+20)$. Then $0.02x = 11.6$, giving $x = 580$.

The number of students who now like to dance is $0.4 \cdot 580 + 20 = 232 + 20 = 252$.

2. Square $AIME$ has sides of length 10 units. Isosceles triangle GEM has base $\overline{EM}$, and the area common to triangle GEM and square $AIME$ is 80 square units. Find the length of the altitude to $\overline{EM}$ in $\triangle GEM$.

Solution:

Here $\overline{EM}$ is a side of the square. Let h be the altitude of triangle GEM. If $h \le 10$, the triangle would lie entirely inside the square, and its area $\frac{1}{2} \cdot 10 \cdot h = 80$ would force $h = 16$, a contradiction. So $h > 10$ and the apex G lies outside the square; the opposite side $\overline{AI}$ cuts off a smaller triangle similar to GEM with height $h-10$ and base $\frac{10(h-10)}{h}$.

The common region is triangle GEM minus that small triangle:

$$80 = 5h - \frac{1}{2} \cdot \frac{10(h-10)}{h} \cdot (h-10) = 5h - \frac{5(h-10)^2}{h}.$$

Multiplying by h gives $80h = 5h^2 - 5(h-10)^2 = 5(20h-100) = 100h - 500$, so $20h = 500$ and $h = 25$.

3. Ed and Sue bike at equal and constant rates. Similarly, they jog at equal and constant rates, and they swim at equal and constant rates. Ed covers 74 kilometers after biking for 2 hours, jogging for 3 hours, and swimming for 4 hours, while Sue covers 91 kilometers after jogging for 2 hours, swimming for 3 hours, and biking for 4 hours. Their biking, jogging, and swimming rates are all whole numbers of kilometers per hour. Find the sum of the squares of Ed's biking, jogging, and swimming rates.

Solution:

Let b, j, and s be the biking, jogging, and swimming rates. The two trips give

$$2b + 3j + 4s = 74 \qquad \text{and} \qquad 4b + 2j + 3s = 91.$$

Doubling the first equation and subtracting the second yields $4j + 5s = 57$, whose positive integer solutions are $(j, s) = (13, 1), (8, 5)$, and $(3, 9)$.

The corresponding values of $2b = 74 - 3j - 4s$ are $31, 30$, and 29, so only $(j, s) = (8, 5)$ gives a whole-number rate, $b = 15$. The sum of the squares is $15^2 + 8^2 + 5^2 = 225 + 64 + 25 = 314$.

4. There exist unique positive integers x and y that satisfy the equation $x^2 + 84x + 2008 = y^2$. Find $x + y$.

Solution:

Completing the square, $x^2 + 84x + 2008 = (x + 42)^2 + 244$, so $y^2 - (x + 42)^2 = 244$, which factors as

$$(y - x - 42)(y + x + 42) = 244 = 2^2 \cdot 61.$$

The two factors have the same parity, and their product is even, so both are even: $y - x - 42 = 2$ and $y + x + 42 = 122$.

Adding gives $y = 62$, and then $x = 18$; indeed $18^2 + 84 \cdot 18 + 2008 = 3844 = 62^2$. Therefore $x + y = 18 + 62 = 80$.

5. A right circular cone has base radius r and height h. The cone lies on its side on a flat table. As the cone rolls on the surface of the table without slipping, the point where the cone's base meets the table traces a circular arc centered at the point where the vertex touches the table. The cone first returns to its original position on the table after making 17 complete rotations. The value of h/r can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

Solution:

The contact point of the base stays at distance $\ell = \sqrt{r^2+h^2}$ (the slant height) from the fixed vertex, so it traces a circle of radius ℓ. Rolling without slipping, the cone makes one rotation for each base circumference of arc, so returning after exactly 17 rotations means

$$2\pi\sqrt{r^2+h^2} = 17 \cdot 2\pi r, \qquad \text{i.e.} \qquad \sqrt{r^2+h^2} = 17r.$$

Squaring gives $h^2 = 288r^2$, so $h/r = \sqrt{288} = 12\sqrt{2}$, and $m+n = 12+2 = 14$.

6. A triangular array of numbers has a first row consisting of the odd integers $1, 3, 5, \ldots, 99$ in increasing order. Each row below the first has one fewer entry than the row above it, and the bottom row has a single entry. Each entry in any row after the top row equals the sum of the two entries diagonally above it in the row immediately above it. How many entries in the array are multiples of 67?

$$\begin{array}{ccccccccccc} 1 & & 3 & & 5 & & \cdots & & 97 & & 99 \\ & 4 & & 8 & & 12 & & \cdots & & 196 & \\ & & & & & \vdots & & & & & \end{array}$$

Solution:

By induction, the nth entry of row r is $2^{r-1}(r+2n-2)$: row 1 gives $2^0(2n-1)$, and summing two adjacent entries of row r gives $2^{r-1}(r+2n-2) + 2^{r-1}(r+2n) = 2^r\big((r+1)+2n-2\big)$, the formula for row $r+1$. Row r has $51-r$ entries, so $1 \le n \le 51-r$.

Since 67 is odd, an entry is a multiple of 67 exactly when $67 \mid r+2n-2$. As n runs through row r, the quantity $r+2n-2$ takes the values $r, r+2, \ldots, 100-r$, all with the same parity as r and all less than 134. So the only possible multiple of 67 is 67 itself, which requires r odd and $r \le 67 \le 100-r$, that is, $r \le 33$.

Each odd row $r = 1, 3, \ldots, 33$ contains exactly one such entry, for a total of 17.

7. Let S_i be the set of all integers n such that $100i \le n < 100(i+1)$. For example, S_4 is the set $\{400, 401, 402, \ldots, 499\}$. How many of the sets $S_0, S_1, S_2, \ldots, S_{999}$ do not contain a perfect square?

Solution:

Consecutive squares a^2 and $(a+1)^2$ differ by $2a+1 \le 99$ for $a \le 49$, so the squares from 1^2 to $50^2 = 2500$ never skip a hundred-block: every set $S_0, S_1, \ldots, S_{25}$ contains a perfect square. For $a \ge 50$ the gap $2a+1 \ge 101$ exceeds 100, so each of the sets $S_{26}, \ldots, S_{999}$ contains at most one square.

The largest number involved is 99999, and $316^2 = 99856 \le 99999 < 317^2$. So the squares landing in $S_{26}, \ldots, S_{999}$ are $51^2, 52^2, \ldots, 316^2$ — that is, 266 squares occupying 266 distinct sets out of those 974.

Therefore $974 - 266 = 708$ sets contain no perfect square.

8. Find the positive integer n such that

$$\arctan\frac{1}{3} + \arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{n} = \frac{\pi}{4}.$$

Solution:

For positive x, y with $xy < 1$, the tangent addition formula gives $\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$. Applying it twice:

$$\arctan\frac{1}{3} + \arctan\frac{1}{4} = \arctan\frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{12}} = \arctan\frac{7}{11}, \qquad \arctan\frac{7}{11} + \arctan\frac{1}{5} = \arctan\frac{\frac{7}{11}+\frac{1}{5}}{1-\frac{7}{55}} = \arctan\frac{23}{24}.$$

The equation becomes $\arctan\frac{23}{24} + \arctan\frac{1}{n} = \arctan 1$, so $\frac{23/24+1/n}{1-23/(24n)} = 1$. Clearing denominators, $23n + 24 = 24n - 23$, giving $n = 47$.

9. Ten identical crates each have dimensions 3 ft $\times$ 4 ft $\times$ 6 ft. The first crate is placed flat on the floor. Each of the remaining nine crates is placed, in turn, flat on top of the previous crate, and the orientation of each crate is chosen at random. Let $\frac{m}{n}$ be the probability that the stack of crates is exactly 41 ft tall, where m and n are relatively prime positive integers. Find m.

Solution:

Each crate independently contributes height $3, 4,$ or 6, each with probability $\frac{1}{3}$, so there are 3^{10} equally likely stacks. If x, y, z crates have heights $3, 4, 6$, then $x + y + z = 10$ and $3x + 4y + 6z = 41$; subtracting three times the first equation gives $y + 3z = 11$, so

$$(x, y, z) = (1, 8, 1), \quad (3, 5, 2), \quad (5, 2, 3).$$

These can be ordered in $\frac{10!}{1!\,8!\,1!} = 90$, $\frac{10!}{3!\,5!\,2!} = 2520$, and $\frac{10!}{5!\,2!\,3!} = 2520$ ways, for 5130 stacks in all. The probability is $\frac{5130}{3^{10}} = \frac{190}{3^7}$, which is in lowest terms since $190 = 2 \cdot 5 \cdot 19$. Thus $m = 190$.

10. Let $ABCD$ be an isosceles trapezoid with $\overline{AD} \parallel \overline{BC}$ whose angle at the longer base $\overline{AD}$ is $\frac{\pi}{3}$. The diagonals have length $10\sqrt{21}$, and point E is at distances $10\sqrt{7}$ and $30\sqrt{7}$ from vertices A and D, respectively. Let F be the foot of the altitude from C to $\overline{AD}$. The distance EF can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Solution:

By the triangle inequality, $30\sqrt{7} = DE \le DA + AE = DA + 10\sqrt{7}$, so $DA \ge 20\sqrt{7}$. On the other hand, in triangle ACD the angle at D is $\frac{\pi}{3}$ and $AC = 10\sqrt{21}$, so the Law of Sines gives

$$DA = \frac{AC \sin \angle DCA}{\sin \frac{\pi}{3}} = \frac{10\sqrt{21}}{\sqrt{3}/2} \sin \angle DCA = 20\sqrt{7} \sin \angle DCA \le 20\sqrt{7}.$$

Both bounds force $DA = 20\sqrt{7}$, so $\angle DCA = 90^\circ$, and equality in the triangle inequality means E lies on line AD with A between D and E. From the right triangle, $DC = \sqrt{DA^2 - AC^2} = \sqrt{2800 - 2100} = 10\sqrt{7}$, and since $\angle CDF = 60^\circ$, the foot satisfies $DF = DC \cos 60^\circ = 5\sqrt{7}$.

Points F and E are on line AD on the same side of D, so $EF = DE - DF = 30\sqrt{7} - 5\sqrt{7} = 25\sqrt{7}$, and $m + n = 25 + 7 = 32$.

11. Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

Solution:

Let a_n and b_n count valid sequences of length n beginning with A and with B. A sequence beginning with A starts with AA followed by any valid sequence of length $n-2$ (possibly empty), so $a_{n+2} = a_n + b_n$, where the empty sequence counts once. A sequence beginning with B starts either with a single B followed by a sequence beginning with A, or with BB followed by a sequence beginning with B, so $b_{n+2} = a_{n+1} + b_n$.

Starting from $(a_1, b_1) = (0, 1)$ and $(a_2, b_2) = (1, 0)$, the pairs (a_n, b_n) for $n = 3, 4, \ldots, 14$ are

$$(1,2),\ (1,1),\ (3,3),\ (2,4),\ (6,5),\ (6,10),\ (11,11),\ (16,21),\ (22,27),\ (37,43),\ (49,64),\ (80,92).$$

The number of valid sequences of length 14 is $80 + 92 = 172$.

12. On a long straight stretch of one-way single-lane highway, cars all travel at the same speed and all obey the safety rule: the distance from the back of the car ahead to the front of the car behind is exactly one car length for each 15 kilometers per hour of speed or fraction thereof. (Thus the front of a car traveling 52 kilometers per hour will be four car lengths behind the back of the car in front of it.)

A photoelectric eye by the side of the road counts the number of cars that pass in one hour. Assuming that each car is 4 meters long and that the cars can travel at any speed, let M be the maximum whole number of cars that can pass the photoelectric eye in one hour. Find the quotient when M is divided by 10.

Solution:

Suppose the cars travel at s kilometers per hour. The gap is $\lceil s/15 \rceil$ car lengths, so successive fronts are $4\lceil s/15 \rceil + 4$ meters apart, and in one hour a column of $1000s$ meters of traffic passes the eye — that is,

$$N = \frac{1000s}{4\lceil s/15 \rceil + 4} = \frac{250s}{\lceil s/15 \rceil + 1}$$

gaps per hour.

For a fixed value $k = \lceil s/15 \rceil$, the count N is largest at $s = 15k$, where it equals $\frac{3750k}{k+1}$. This is always less than 3750 but approaches 3750 as k grows. Although the gap count never reaches 3750, the car count can: choose k so large that more than 3749 gaps pass, and start the hour with a car exactly at the eye. That car, plus one car for each of the 3749 complete gaps that follow, makes 3750 cars.

So $M = 3750$, and the quotient when M is divided by 10 is 375.

13. Let

$$p(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3.$$

Suppose that

$$p(0,0) = p(1,0) = p(-1,0) = p(0,1) = p(0,-1) = p(1,1) = p(1,-1) = p(2,2) = 0.$$

There is a point $\left(\frac{a}{c}, \frac{b}{c}\right)$ for which $p\left(\frac{a}{c}, \frac{b}{c}\right) = 0$ for all such polynomials, where $a, b,$ and c are positive integers, a and c are relatively prime, and $c > 1$. Find $a + b + c$.

Solution:

From $p(0,0) = 0$ we get $a_0 = 0$. Adding and subtracting $p(1,0) = p(-1,0) = 0$ gives $a_3 = 0$ and $a_6 = -a_1$; similarly $p(0, \pm 1) = 0$ give $a_5 = 0$ and $a_9 = -a_2$. Then $p(1,1) = 0$ and $p(1,-1) = 0$ reduce to $a_4 + a_7 + a_8 = 0$ and $-a_4 - a_7 + a_8 = 0$, so $a_8 = 0$ and $a_7 = -a_4$. Now $p = a_1(x - x^3) + a_2(y - y^3) + a_4(xy - x^2y)$, and $p(2,2) = 0$ gives $-6a_1 - 6a_2 - 4a_4 = 0$, i.e. $a_4 = -\frac{3}{2}(a_1 + a_2)$.

Therefore

$$p = a_1\left[x - x^3 - \tfrac{3}{2}xy(1-x)\right] + a_2\left[y - y^3 - \tfrac{3}{2}xy(1-x)\right],$$

and a point (r, s) that is a zero for every choice of a_1, a_2 must kill both brackets. The first bracket factors as $r(1 - r)\left(1 + r - \frac{3}{2}s\right)$, so for a new point (with $r \neq 0, 1$) we need $s = \frac{2}{3}(r+1)$. The second bracket is $\frac{1}{2}s(2 - 2s^2 - 3r + 3r^2)$; substituting $s^2 = \frac{4}{9}(r+1)^2$ turns $2 - 2s^2 - 3r + 3r^2 = 0$ into

$$\frac{19r^2 - 43r + 10}{9} = 0,$$

whose roots are $r = 2$ and $r = \frac{5}{19}$.

The root $r = 2$ reproduces the given point $(2, 2)$, so the new point has $r = \frac{5}{19}$ and $s = \frac{2}{3} \cdot \frac{24}{19} = \frac{16}{19}$. Thus $(a, b, c) = (5, 16, 19)$ and $a + b + c = 40$.

14. Let $\overline{AB}$ be a diameter of circle ω. Extend $\overline{AB}$ through A to C. Point T lies on ω so that line CT is tangent to ω. Point P is the foot of the perpendicular from A to line CT. Suppose $AB = 18$, and let m denote the maximum possible length of segment BP. Find m^2.

Solution:

Place the center O at the origin with radius 9, so $A = (-9, 0)$ and $B = (9, 0)$. If the point of tangency is $T = (9\cos t, 9\sin t)$, the tangent line is $x\cos t + y\sin t = 9$; it meets the x-axis at $C = (9/\cos t, 0)$, which lies beyond A exactly when $-1 < \cos t < 0$. Writing $u = \cos t$, the signed distance from A to the line is $-9u - 9$, so the foot of the perpendicular is $P = A + 9(1+u)(\cos t, \sin t)$.

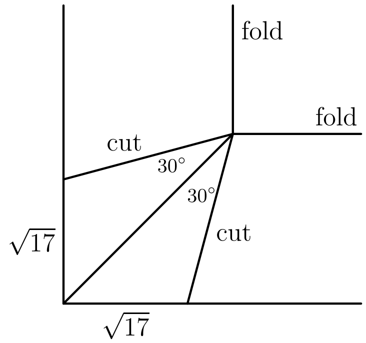
Then $P - B = \big(9(u^2 + u - 2),\ 9(1+u)\sin t\big)$, and using $\sin^2 t = 1 - u^2$:

$$\frac{BP^2}{81} = (u^2 + u - 2)^2 + (1+u)^2(1-u^2) = 5 - 2u - 3u^2.$$

This quadratic in u is maximized at $u = -\frac{1}{3}$, which is inside $(-1, 0)$ (there $C = (-27, 0)$), giving $\frac{BP^2}{81} = 5 + \frac{2}{3} - \frac{1}{3} = \frac{16}{3}$.

Therefore $m^2 = 81 \cdot \frac{16}{3} = 432$.

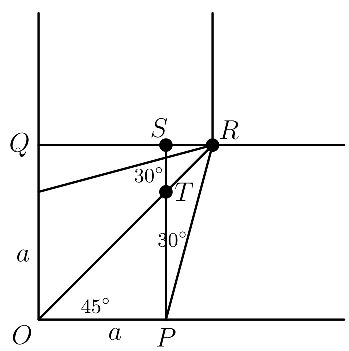
15. A square piece of paper has sides of length 100. From each corner a wedge is cut in the following manner: at each corner, the two cuts for the wedge each start at distance $\sqrt{17}$ from the corner, and they meet on the diagonal at an angle of 60° (see the figure below). The paper is then folded up along the lines joining the vertices of adjacent cuts. When the two edges of a cut meet, they are taped together. The result is a paper tray whose sides are not at right angles to the base. The height of the tray, that is, the perpendicular distance between the plane of the base and the plane formed by the upper edges, can be written in the form $\sqrt[n]{m}$, where m and n are positive integers, $m < 1000$, and m is not divisible by the nth power of any prime. Find $m+n$.



Solution:

Put the corner at the origin O with the two sides along the positive axes, and write $a=\sqrt{17}$. The cut on the bottom edge starts at $P=(a,0)$, and the two cuts meet at R on the diagonal $y=x$, each making a 30° angle with the diagonal. In triangle OPR, $\angle ROP = 45^\circ$ and $\angle ORP = 30^\circ$, so the Law of Sines gives $PR = \frac{OP\sin 45^\circ}{\sin 30^\circ} = a\sqrt{2}$. The fold lines are the horizontal and vertical lines through R. Let S be the point of the horizontal fold line directly above P, and $T=(a,a)$ the point where the vertical line through P meets the diagonal. Since $\angle OPR = 105^\circ$, segment PR makes a 75° angle with the bottom edge, so

$$SP = PR\sin 75^\circ = a\sqrt{2}\cdot\frac{\sqrt{6}+\sqrt{2}}{4} = a\,\frac{\sqrt{3}+1}{2}, \qquad ST = SP - PT = a\,\frac{\sqrt{3}+1}{2} - a = a\,\frac{\sqrt{3}-1}{2}.$$



When the bottom strip folds up along the horizontal line through R, point P stays at distance SP from S, moving in the vertical plane through P perpendicular to that fold line. By symmetry the two taped cut edges meet above the diagonal, so P lands at a point P' directly above T, and $P'T$ is the height of the tray. By the Pythagorean theorem,

$$P'T^2 = P'S^2 - ST^2 = a^2\left(\frac{\sqrt{3}+1}{2}\right)^2 - a^2\left(\frac{\sqrt{3}-1}{2}\right)^2 = a^2\sqrt{3}.$$

So the height is $a\cdot 3^{1/4} = \sqrt{17}\cdot\sqrt[4]{3} = \sqrt[4]{17^2\cdot 3} = \sqrt[4]{867}$, and $m+n = 867+4 = 871$.

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