2007 AIME I Solutions

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1. How many positive perfect squares less than 10^6 are multiples of 24?

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Solution:

Since $24 = 2^3 \cdot 3$, a square N^2 is a multiple of 24 exactly when N is a multiple of 12 : the factor 2^3 forces N to contain 2^2, the factor 3 forces N to contain 3, and conversely $(12m)^2 = 144m^2$ is always a multiple of 24. Also $N^2 < 10^6$ exactly when $N < 1000$.

The multiples of 12 less than 1000 are $12, 24, \ldots, 996$, and there are $\frac{996}{12} = 83$ of them.

2. A 100 foot long moving walkway moves at a constant rate of 6 feet per second. Al steps onto the start of the walkway and stands. Bob steps onto the start of the walkway two seconds later and strolls forward along the walkway at a constant rate of 4 feet per second. Two seconds after that, Cy reaches the start of the walkway and walks briskly forward beside the walkway at a constant rate of 8 feet per second. At a certain time, one of these three persons is exactly halfway between the other two. At that time, find the distance in feet between the start of the walkway and the middle person.

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Solution:

Measure time t in seconds from when Al steps on. Al stands on the walkway, so he is at $6t$; Bob moves at $6+4=10$ feet per second, so he is at $10(t-2)$; Cy walks beside the walkway at 8 feet per second, so he is at $8(t-4)$. All three are moving once $t \geq 4$.

The middle person's position doubled must equal the sum of the other two. If Bob were in the middle, $20(t-2)=6t+8(t-4)$ gives $t=\frac{4}{3}<4$, impossible. If Cy were in the middle, $16(t-4)=6t+10(t-2)$ reduces to $-64=-20$, with no solution. If Al is in the middle, $12t=10(t-2)+8(t-4)=18t-52$, so $t=\frac{26}{3}$.

At that moment Al is at $6 \cdot \frac{26}{3}=52$ feet, while Bob and Cy are at $\frac{200}{3}$ and $\frac{112}{3}$, whose average is indeed 52. The middle person is 52 feet from the start.

3. The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.

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Solution:

By the binomial theorem, $z^2 = (81 - b^2) + 18bi$ and $z^3 = (729 - 27b^2) + (243b - b^3)i$. Setting the imaginary parts equal gives $18b = 243b - b^3$.

Since b is positive we may divide by b, leaving $b^2 = 243 - 18 = 225$, so $b = 15$.

4. Three planets revolve about a star in coplanar circular orbits with the star at the center. All planets revolve in the same direction, each at a constant speed, and the periods of their orbits are $60, 84,$ and 140 years. The positions of the star and all three planets are currently collinear. They will next be collinear after n years. Find n.

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Solution:

All four bodies lie on one line exactly when every pair of planets is collinear with the star, i.e. when each pair's angular positions differ by a multiple of 180° — half a revolution. In n years the planets complete $\frac{n}{60}$, $\frac{n}{84}$, and $\frac{n}{140}$ revolutions, so the pairwise differences are

$$\frac{n}{60} - \frac{n}{84} = \frac{n}{210}, \qquad \frac{n}{84} - \frac{n}{140} = \frac{n}{210}, \qquad \frac{n}{60} - \frac{n}{140} = \frac{n}{105}.$$

We need $\frac{n}{210}$ and $\frac{n}{105}$ to be multiples of $\frac{1}{2}$. The first requires n to be a multiple of 105, and any such n makes $\frac{n}{105}$ an integer. The smallest positive choice is $n = 105$.

5. The formula for converting a Fahrenheit temperature F to the corresponding Celsius temperature C is $C = \frac{5}{9}(F - 32)$. An integer Fahrenheit temperature is converted to Celsius and rounded to the nearest integer; the resulting integer Celsius temperature is converted back to Fahrenheit and rounded to the nearest integer. For how many integer Fahrenheit temperatures T with $32 \leq T \leq 1000$ does the original temperature equal the final temperature?

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Solution:

Adding 9 to F adds exactly 5 to $\frac{5}{9}(F - 32)$, hence 5 to the rounded Celsius value, hence 9 to the final Fahrenheit value. So T returns to itself if and only if $T + 9$ does, and it suffices to check nine consecutive temperatures. Checking 32 through 40 : the final values are $32, 34, 34, 36, 36, 37, 37, 39, 39,$ so exactly the five temperatures $32, 34, 36, 37, 39$ survive.

The range from 32 through 994 contains $963 = 107 \cdot 9$ integers, contributing $107 \cdot 5 = 535$ survivors. The remaining $995, \ldots, 1000$ behave like $32, \ldots, 37,$ of which $32, 34, 36, 37$ survive, adding 4 more.

The total is $535 + 4 = 539.$

6. A frog is placed at the origin on the number line, and moves according to the following rule: in a given move, the frog advances to either the closest point with a greater integer coordinate that is a multiple of 3, or to the closest point with a greater integer coordinate that is a multiple of 13. A *move sequence* is a sequence of coordinates which correspond to valid moves, beginning with 0, and ending with 39. For example, $0, 3, 6, 13, 15, 26, 39$ is a move sequence. How many move sequences are possible for the frog?

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Solution:

Split the journey at the landmarks 13 and 26. From 0 the frog climbs the multiples of 3 and may jump to 13 from any of $0, 3, 6, 9, 12$, giving 5 routes from 0 to 13; likewise there are 5 routes from 13 to 26 (jump to 26 from $13, 15, 18, 21, 24$) and 5 from 26 to 39. To skip 13 entirely the frog must take the multiple-of-3 option every time through $12 \to 15$, then jump to 26 from one of $15, 18, 21, 24:4$ routes from 0 to 26 avoiding 13. Similarly there are 4 routes from 13 to 39 avoiding 26, and 4 from 0 to 39 avoiding both.

Combining the segments: through both landmarks, $5 \cdot 5 \cdot 5 = 125$; through 13 only, $5 \cdot 4 = 20$; through 26 only, $4 \cdot 5 = 20$; through neither, 4. The total is $125 + 20 + 20 + 4 = 169$.

7. Let

$$N = \sum_{k=1}^{1000} k \left(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor \right).$$

Find the remainder when N is divided by 1000. (Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, and $\lceil x \rceil$ denotes the least integer that is greater than or equal to x.)

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Solution:

The difference $\lceil x \rceil - \lfloor x \rfloor$ equals 1 when x is not an integer and 0 when it is. Now $\log_{\sqrt{2}} k$ is an integer exactly when $k = (\sqrt{2})^j$ for some integer j, and for k to be an integer, j must be even — that is, k must be a power of 2. The powers at most 1000 are $2^0, 2^1, \ldots, 2^9 = 512.$

Therefore

$$N = \sum_{k=1}^{1000} k - \sum_{j=0}^{9} 2^j = \frac{1000 \cdot 1001}{2} - 1023 = 500500 - 1023 = 499477,$$

and the remainder upon division by 1000 is 477.

8. The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 + (2k-43)x + k$ are both factors of $P(x)$?

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Solution:

If Q_1 and Q_2 had no common root, their product — of degree 4 — would divide the cubic $P(x)$, which is impossible. So they share a root r, and $2Q_1(r) - Q_2(r) = 0$. Computing, $2Q_1(x) - Q_2(x) = -15x - 3k$, so $r = -\frac{k}{5}$.

Substituting into $Q_1(r) = 0$ gives $\frac{k^2}{25} - (k-29)\frac{k}{5} - k = 0$; multiplying by 25 and simplifying yields $-4k^2 + 120k = 0$, so $k = 0$ or $k = 30$.

For $k = 30$, $Q_1(x) = x^2 + x - 30 = (x+6)(x-5)$ and $Q_2(x) = 2x^2 + 17x + 30 = (x+6)(2x+5)$, and both divide $P(x) = (x+6)(x-5)(2x+5)$. The largest value is 30.

9. In right triangle ABC with right angle C, $CA = 30$ and $CB = 16$. Its legs $\overline{CA}$ and $\overline{CB}$ are extended beyond A and B. Points O_1 and O_2 lie in the exterior of the triangle and are the centers of two circles with equal radii. The circle with center O_1 is tangent to the hypotenuse and to the extension of leg CA, the circle with center O_2 is tangent to the hypotenuse and to the extension of leg CB, and the circles are externally tangent to each other. The length of the radius of either circle can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

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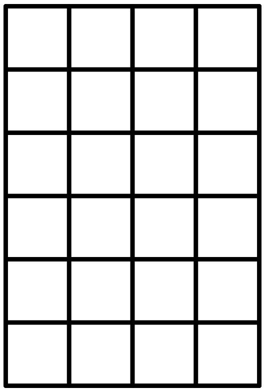
Solution:

The hypotenuse is $AB = \sqrt{30^2 + 16^2} = 34$. Let T_1 and T_2 be the points where the circles touch AB. Both centers lie at distance r from line AB on the side away from the triangle, so $\overline{O_1O_2}$ is parallel to AB and $T_1T_2 = O_1O_2 = 2r$, since the circles are externally tangent. Thus $AB = AT_1 + 2r + T_2B$.

Circle O_1 is inscribed in the angle at A between ray AB and the extension of $\overline{CA}$ beyond A, which measures $180^\circ - \angle A$. Its tangent length from A is therefore $AT_1 = r\big/\tan\left(90^\circ - \frac{A}{2}\right) = r\tan\frac{A}{2}$. With $\sin A = \frac{16}{34}$ and $\cos A = \frac{30}{34}$, the half-angle formula gives $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A} = \frac{16}{64} = \frac{1}{4}$, and similarly $\tan\frac{B}{2} = \frac{30}{34+16} = \frac{3}{5}$. So $34 = \frac{r}{4} + 2r + \frac{3r}{5} = \frac{57r}{20}$, giving $r = \frac{680}{57}$. Since $680 = 2^3 \cdot 5 \cdot 17$ and $57 = 3 \cdot 19$ share no common factor, $p + q = 680 + 57 = 737$.

10. In the 6×4 grid shown, 12 of the 24 squares are to be shaded so that there are two shaded squares in each row and three shaded squares in each column. Let N be the number of shadings with this property. Find the remainder when N is divided by $1000.$



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Solution:

Shade three of the six rows in column $1: \binom{6}{3} = 20$ ways. Let k be the number of rows shaded in both columns 1 and 2; column 2 can then be chosen in $\binom{3}{k}\binom{3}{3-k}$ ways. After these two columns, k rows are complete with two shaded squares, $6-2k$ rows have one, and k rows have none.

The empty rows must be shaded in both columns 3 and $4.$ Column 3 takes those k rows plus $3-k$ of the $6-2k$ singly-shaded rows, in $\binom{6-2k}{3-k}$ ways, and column 4 is then forced: it must cover the empty rows and exactly the singly-shaded rows skipped by column $3.$

Summing,

$$N = 20\sum_{k=0}^{3}\binom{3}{k}\binom{3}{3-k}\binom{6-2k}{3-k} = 20(20+54+18+1) = 1860,$$

so the remainder is $860.$

11. For each positive integer p, let $b(p)$ denote the unique positive integer k such that $|k-\sqrt{p}|<\frac{1}{2}$. For example, $b(6)=2$ and $b(23)=5$. If $S=\sum_{p=1}^{2007} b(p)$, find the remainder when S is divided by 1000.

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Solution:

For a positive integer k, the condition $|k-\sqrt{p}|<\frac{1}{2}$ means $\left(k-\frac{1}{2}\right)^2<p<\left(k+\frac{1}{2}\right)^2$, which for integers p is exactly $k^2-k+1\leq p\leq k^2+k$. So $b(p)=k$ for precisely $2k$ values of p.

Since $44^2+44=1980$, the blocks $k=1,\ldots,44$ exactly cover $p\leq 1980$ and contribute

$$\sum_{k=1}^{44} k\cdot 2k=2\cdot\frac{44\cdot 45\cdot 89}{6}=58740.$$

The remaining 27 values $p=1981,\ldots,2007$ each have $b(p)=45$, adding $27\cdot 45=1215$.

Thus $S=58740+1215=59955$, and the remainder is 955.

12. In isosceles triangle ABC, A is located at the origin and B is located at $(20,0)$. Point C is in the first quadrant with $AC = BC$ and $\angle BAC = 75^\circ$. If $\triangle ABC$ is rotated counterclockwise about point A until the image of C lies on the positive y-axis, the area of the region common to the original triangle and the rotated triangle is in the form $p\sqrt{2} + q\sqrt{3} + r\sqrt{6} + s$, where p, q, r, s are integers. Find $\frac{p-q+r-s}{2}$.

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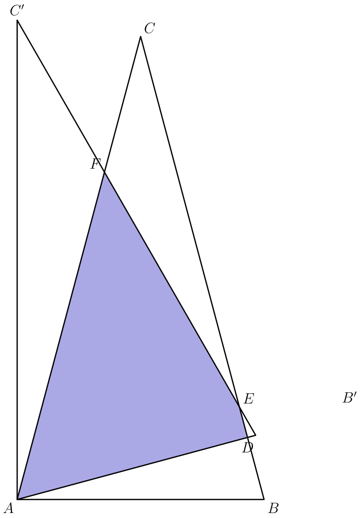
Solution:



Since AC makes a 75° angle with the positive x-axis, the rotation is by 15°. Let B' and C' be the images of B and C. Because $\angle B'AB = 15^\circ$ and $\angle ABC = 75^\circ$, segment AB' is perpendicular to BC; let D be their intersection, and let $E = BC \cap B'C'$ and $F = AC \cap B'C'$. The common region is the quadrilateral $ADEF$, whose area is $[AB'F] - [EB'D]$.

In triangle $AB'F$, $\angle FAB' = 75^\circ - 15^\circ = 60^\circ$ and $\angle AB'F = 75^\circ$, so $\angle AFB' = 45^\circ$, and the law of sines gives $B'F = 20\sin 60^\circ / \sin 45^\circ = 10\sqrt{6}$. With $\sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$,

$$[AB'F] = \tfrac{1}{2} \cdot 20 \cdot 10\sqrt{6}\,\sin 75^\circ = 50(3+\sqrt{3}).$$

In right triangle ABD, $AD = 20\cos 15^\circ$ and $BD = 20\sin 15^\circ$, so $[ABD] = 200\sin 15^\circ \cos 15^\circ = 100\sin 30^\circ = 50$, and $B'D = 20(1-\cos 15^\circ)$. Triangles $EB'D$ and ABD are similar (right angles at D, and $\angle EB'D = \angle ABD = 75^\circ$), so, using $\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$,

$$[EB'D] = 50\left(\frac{1-\cos 15^\circ}{\sin 15^\circ}\right)^2 = 50\left(15+8\sqrt{3}-6\sqrt{6}-10\sqrt{2}\right).$$

Therefore $[ADEF] = 50(3+\sqrt{3}) - 50(15+8\sqrt{3}-6\sqrt{6}-10\sqrt{2}) = 500\sqrt{2} - 350\sqrt{3} + 300\sqrt{6} - 600$, so $(p,q,r,s) = (500,-350,300,-600)$ and $\frac{p-q+r-s}{2} = \frac{1750}{2} = 875$.

13. A square pyramid with base $ABCD$ and vertex E has eight edges of length 4. A plane passes through the midpoints of $\overline{AE}$, $\overline{BC}$, and $\overline{CD}$. The plane's intersection with the pyramid has an area that can be expressed as $\sqrt{p}$. Find p.

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Solution:

Place the base at $A=(0,0,0), B=(4,0,0), C=(4,4,0), D=(0,4,0)$; the apex is then $E=(2,2,2\sqrt{2})$, since $2^2+2^2+8=16$. The given midpoints are $R=(1,1,\sqrt{2}), S=(4,2,0)$, and $T=(2,4,0)$, and all three satisfy $x+y+2\sqrt{2}\,z=6$, the equation of the cutting plane.

Parametrizing edges $\overline{BE}$ and $\overline{DE}$ shows the plane meets them at $U=\left(\frac{7}{2},\frac{1}{2},\frac{\sqrt{2}}{2}\right)$ and $V=\left(\frac{1}{2},\frac{7}{2},\frac{\sqrt{2}}{2}\right)$. The cross-section is the pentagon $RUSTV$ with $RU=RV=\sqrt{7}, US=VT=\sqrt{3}, ST=2\sqrt{2}$, and diagonal $UV=3\sqrt{2}$.

Split the pentagon along $\overline{UV}$. Isosceles triangle RUV has height $\sqrt{7-\frac{9}{2}}=\sqrt{\frac{5}{2}}$ and area $\frac{1}{2}\cdot 3\sqrt{2}\cdot\sqrt{\frac{5}{2}}=\frac{3\sqrt{5}}{2}$. Isosceles trapezoid $USTV$ has height $\sqrt{3-\frac{1}{2}}=\sqrt{\frac{5}{2}}$ and area $\frac{1}{2}(3\sqrt{2}+2\sqrt{2})\sqrt{\frac{5}{2}}=\frac{5\sqrt{5}}{2}$. The total is $4\sqrt{5}=\sqrt{80}$, so $p=80$.

14. Let a sequence be defined as follows: $a_1 = 3, a_2 = 3$, and for $n \geq 2, a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the largest integer less than or equal to $\frac{a_{2007}^2+a_{2006}^2}{a_{2007}a_{2006}}$.

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Solution:

For $n \geq 3$, both $a_{n+1}a_{n-1} = a_n^2 + 2007$ and $a_n a_{n-2} = a_{n-1}^2 + 2007$ hold. Subtracting and regrouping gives $a_{n-1}(a_{n+1} + a_{n-1}) = a_n(a_n + a_{n-2})$, so $\frac{a_{n+1}+a_{n-1}}{a_n}$ has the same value for every $n \geq 2$. Since $a_3 = \frac{3^2+2007}{3} = 672$, that value is $\frac{672+3}{3} = 225$, and the sequence satisfies $a_{n+1} = 225a_n - a_{n-1}$.

Multiplying $a_{n+1} + a_{n-1} = 225a_n$ by a_{n+1} and substituting $a_{n+1}a_{n-1} = a_n^2 + 2007$ yields $a_{n+1}^2 + a_n^2 + 2007 = 225\, a_n a_{n+1}$, so

$$\frac{a_{n+1}^2 + a_n^2}{a_{n+1}a_n} = 225 - \frac{2007}{a_n a_{n+1}}.$$

The sequence increases: $a_3 = 672 > a_2$, and $a_{n+1} = 225a_n - a_{n-1} > a_n$ whenever $a_n > a_{n-1}$. Hence $a_{2006}a_{2007} > 672^2 > 2007$, so the fraction lies strictly between 224 and 225, and the answer is 224.

15. Let ABC be an equilateral triangle, and let D and F be points on sides BC and AB, respectively, with $FA = 5$ and $CD = 2$. Point E lies on side CA such that $\angle DEF = 60^\circ$. The area of triangle DEF is $14\sqrt{3}$. The two possible values of the length of side AB are $p \pm q\sqrt{r}$, where p and q are rational, and r is an integer not divisible by the square of a prime. Find r.

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Solution:

Let $s = AB$ and $t = AE$. Using the 60° angles at A, B, C and the area formula $\frac{1}{2}xy\sin 60^\circ : [AEF] = \frac{\sqrt{3}}{4} \cdot 5t, [BFD] = \frac{\sqrt{3}}{4}(s-5)(s-2)$, and $[CDE] = \frac{\sqrt{3}}{4} \cdot 2(s-t)$. Subtracting all three from $[ABC] = \frac{\sqrt{3}}{4}s^2$ and simplifying,

$$[DEF] = \frac{\sqrt{3}}{4}\big(5(s-t) + 2t - 10\big) = 14\sqrt{3},$$

so $5(s-t) + 2t = 66$.

At E, the angles $\angle AEF$ and $\angle CED$ sum to $180^\circ - 60^\circ = 120^\circ$, while in triangle AEF the angles $\angle AEF$ and $\angle AFE$ also sum to 120°. Hence $\angle AFE = \angle CED$, and since $\angle A = \angle C = 60^\circ$, triangles AEF and CDE are similar. Then $\frac{AE}{AF} = \frac{CD}{CE}$ gives $\frac{t}{5} = \frac{2}{s-t}$, so $t(s-t) = 10$.

Substituting $s - t = \frac{10}{t}$ into $5(s-t) + 2t = 66$ gives $\frac{50}{t} + 2t = 66$, or $t^2 - 33t + 25 = 0$, so $t = \frac{33 \pm \sqrt{989}}{2}$. From $\frac{25}{t} = 33 - t$ we get $\frac{10}{t} = \frac{2}{5}(33-t)$, so $s = t + \frac{10}{t} = \frac{3t+66}{5} = \frac{231 \pm 3\sqrt{989}}{10}$. Both values yield valid configurations, so $r = 989$.

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