2006 AIME II Solutions

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1. In convex hexagon $ABCDEF$, all six sides are congruent, $\angle A$ and $\angle D$ are right angles, and $\angle B, \angle C, \angle E$, and $\angle F$ are congruent. The area of the hexagonal region is $2116(\sqrt{2}+1)$. Find AB.

Solution:

The angles of a hexagon sum to 720°, so each of the four congruent angles measures $\frac{720-2\cdot 90}{4} = 135$ degrees. Let $AB = x$. The diagonals BF and CE cut off the right isosceles triangles FAB and CDE, each with legs x and hypotenuse $x\sqrt{2}$, and the 135° angles guarantee that the remaining piece $BCEF$ is a rectangle with sides $x\sqrt{2}$ and x.

Hence the area is

$$2 \cdot \frac{1}{2}x^2 + x \cdot x\sqrt{2} = x^2(1+\sqrt{2}) = 2116(\sqrt{2}+1),$$

so $x^2 = 2116$ and $AB = x = 46$.

2. The lengths of the sides of a triangle with positive area are $\log_{10} 12, \log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n.

Solution:

The triangle inequality requires $\log n < \log 12 + \log 75 = \log 900$ and $\log 12 + \log n > \log 75$, that is $\log n > \log 75 - \log 12 = \log \frac{25}{4}$. (The third inequality is the first one again.)

So $\frac{25}{4} < n < 900$, which for integers means $7 \leq n \leq 899$. That gives $899 - 7 + 1 = 893$ possible values of n.

3. Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

Solution:

$P = 1 \cdot 3 \cdot 5 \cdots 199$, so k is the total number of factors of 3 among the odd numbers up to 199. The odd multiples of 3 are $3 \cdot 1, 3 \cdot 3, \ldots, 3 \cdot 65$, and there are 33 of them. The odd multiples of 9 are $9 \cdot 1, \ldots, 9 \cdot 21 : 11$ of them. The odd multiples of 27 are $27, 81, 135, 189 : 4$ of them. The only odd multiple of 81 at most 199 is 81 itself, and there are no multiples of 243.

Each layer contributes one additional factor of 3, so $k = 33 + 11 + 4 + 1 = 49$.

4. Let $(a_1, a_2, a_3, \ldots, a_{12})$ be a permutation of $(1, 2, 3, \ldots, 12)$ for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6 \quad \text{and} \quad a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}.$$

An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.

Solution:

The term a_6 is smaller than every other term of the permutation, so $a_6 = 1$. Now choose which five of the remaining 11 numbers occupy positions 1 through 5 : they must appear in decreasing order, so their arrangement is forced, and the other six numbers must fill positions 7 through 12 in increasing order, which is also forced.

Every choice of the five numbers gives exactly one valid permutation, so the count is $\binom{11}{5} = 462.$

5. When rolling a certain unfair six-sided die with faces numbered $1, 2, 3, 4, 5,$ and $6,$ the probability of obtaining face F is greater than $\frac{1}{6}$, the probability of obtaining the face opposite face F is less than $\frac{1}{6}$, the probability of obtaining each of the other faces is $\frac{1}{6}$, and the sum of the numbers on each pair of opposite faces is $7.$ When two such dice are rolled, the probability of obtaining a sum of 7 is $\frac{47}{288}$. Given that the probability of obtaining face F is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n.$

Solution:

Let the probability of face F be $\frac{1}{6} + x,$ so the face opposite F has probability $\frac{1}{6} - x$ (the six probabilities must sum to 1). Since opposite faces sum to $7,$ a total of 7 occurs exactly when the two dice show a pair of opposite faces. Of the six ordered pairs that sum to $7,$ four use only ordinary faces, and two pair F with its opposite. Thus

$$\frac{47}{288} = 4\left(\frac{1}{6}\right)^2 + 2\left(\frac{1}{6} + x\right)\left(\frac{1}{6} - x\right) = \frac{1}{6} - 2x^2.$$

Since $\frac{1}{6} = \frac{48}{288}$, this gives $2x^2 = \frac{1}{288}$, so $x = \frac{1}{24}$. The probability of face F is $\frac{1}{6} + \frac{1}{24} = \frac{5}{24}$, and $m + n = 5 + 24 = 29.$

6. Square $ABCD$ has sides of length 1. Points E and F are on $\overline{BC}$ and $\overline{CD}$, respectively, so that $\triangle AEF$ is equilateral. A square with vertex B has sides that are parallel to those of $ABCD$ and a vertex on $\overline{AE}$. The length of a side of this smaller square is $\frac{a-\sqrt{b}}{c}$, where $a, b,$ and c are positive integers and b is not divisible by the square of any prime. Find $a+b+c$.

Solution:

Place $A=(0,0), B=(1,0), C=(1,1), D=(0,1)$. By the symmetry of the equilateral triangle across diagonal $\overline{AC}$, we have $BE=DF$. Let $BE=t$, so $CE=CF=1-t$. Then $AE^2=1+t^2$ and $EF^2=2(1-t)^2$, and setting them equal gives $t^2-4t+1=0$, so $t=2-\sqrt{3}$ (taking the root less than 1).

Thus $E=(1,\ 2-\sqrt{3})$, and line AE is $y=(2-\sqrt{3})x$. If the smaller square has side q, its vertex opposite B is $(1-q,\ q)$, which must lie on line $AE:$

$$q=(2-\sqrt{3})(1-q) \quad \Longrightarrow \quad q=\frac{2-\sqrt{3}}{3-\sqrt{3}}=\frac{(2-\sqrt{3})(3+\sqrt{3})}{6}=\frac{3-\sqrt{3}}{6}.$$

So $a=3, b=3, c=6$, and $a+b+c=12$.

7. Find the number of ordered pairs of positive integers (a, b) such that $a+b=1000$ and neither a nor b has a zero digit.

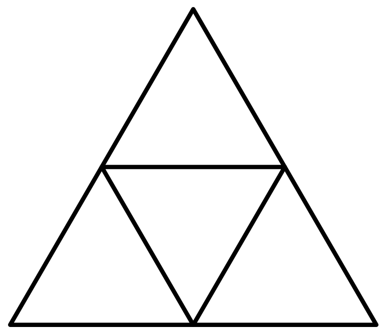
Solution:

There are 999 pairs in all ($a=1, \ldots, 999$); count the forbidden ones. If a has units digit 0, so does b, and writing $a=10r, b=10s$ gives $r+s=100$ with $1 \leq r \leq 99$: that is 99 forbidden pairs.

Now suppose both units digits are nonzero. Then a number in the pair has a zero digit exactly when it is a three-digit number of the form $h0u$ with $h, u \in \{1, \ldots, 9\}$ (a one- or two-digit number with nonzero units digit has no zero digit). If $a=h0u$, then $b=1000-a=100(9-h)+90+(10-u)$ has tens digit 9, so b is not also of that form. Hence the forbidden pairs here are those where exactly one of a, b equals $h0u$: $81+81=162$ pairs.

The total number of forbidden pairs is $99+162=261$, so the answer is $999-261=738.$

8. There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles as shown. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small triangles are of the same color. Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles can be constructed?



Solution:

The rotations and reflections of the large triangle realize every permutation of the three corner triangles while fixing the center triangle. So two large triangles are indistinguishable exactly when they have the same center color and the same multiset of three corner colors.

Count the multisets of corner colors from six colors: all three the same (6 ways), exactly two the same ($6 \cdot 5 = 30$ ways, choosing the repeated color and then the different one), or all three different ($\binom{6}{3} = 20$ ways). That is $6 + 30 + 20 = 56$ multisets.

With 6 independent choices for the center color, the total is $6 \cdot 56 = 336.$

9. Circles $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ have their centers at $(0,0)$, $(12,0)$, and $(24,0)$, and have radii 1, 2, and 4, respectively. Line t_1 is a common internal tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ and has a positive slope, and line t_2 is a common internal tangent to $\mathcal{C}_2$ and $\mathcal{C}_3$ and has a negative slope. Given that lines t_1 and t_2 intersect at (x,y), and that $x = p - q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.

Solution:

A common internal tangent meets the segment between the centers at the point dividing it in the ratio of the radii. For $\mathcal{C}_1$ and $\mathcal{C}_2$ that point is $(4,0)$, at distance 4 from $(0,0)$. If t_1 makes angle θ with the x-axis, then $\sin\theta = \frac{1}{4}$, so $\tan\theta = \frac{1}{\sqrt{15}}$ and t_1 is $y = \frac{1}{\sqrt{15}}(x-4)$. For $\mathcal{C}_2$ and $\mathcal{C}_3$ the point is $(16,0)$, at distance 4 from $(12,0)$; here $\sin\theta = \frac{2}{4} = \frac{1}{2}$, so the slope is $-\frac{1}{\sqrt{3}}$ and t_2 is $y = -\frac{1}{\sqrt{3}}(x-16)$.

Setting the two expressions equal and multiplying by $\sqrt{15}$ gives $x - 4 = -\sqrt{5}\,(x - 16)$, so $x(1+\sqrt{5}) = 4 + 16\sqrt{5}$ and

$$x = \frac{4+16\sqrt{5}}{1+\sqrt{5}} = \frac{(4+16\sqrt{5})(\sqrt{5}-1)}{4} = \frac{76-12\sqrt{5}}{4} = 19 - 3\sqrt{5}.$$

Thus $p + q + r = 19 + 3 + 5 = 27$.

10. Seven teams play a soccer tournament in which each team plays every other team exactly once. No ties occur, each team has a 50% chance of winning each game it plays, and the outcomes of the games are independent. In each game, the winner is awarded 1 point and the loser gets 0 points. The total points are accumulated to decide the ranks of the teams. In the first game of the tournament, team A beats team B. The probability that team A finishes with more points than team B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Teams A and B each have 5 games left, none against each other, so all $2^5 \cdot 2^5 = 1024$ outcomes are equally likely. Since A already leads by one point, A finishes with more points exactly when A wins at least as many remaining games as B does.

The number of outcomes with equal win counts is

$$\sum_{k=0}^{5} \binom{5}{k}^2 = \binom{10}{5} = 252.$$

By symmetry, the other $1024 - 252 = 772$ outcomes split evenly between A winning more and B winning more.

So the probability is $\frac{252+386}{1024} = \frac{638}{1024} = \frac{319}{512}$, and $m+n = 319+512 = 831.$

11. A sequence is defined as follows: $a_1 = a_2 = a_3 = 1$, and, for all positive integers n, $a_{n+3} = a_{n+2} + a_{n+1} + a_n$. Given that $a_{28} = 6090307$, $a_{29} = 11201821$, and $a_{30} = 20603361$, find the remainder when $\sum_{k=1}^{28} a_k$ is divided by 1000.

Solution:

Let $S_n = a_1 + \cdots + a_n$. We claim $2S_n = a_{n+2} + a_n$, which holds for $n = 1$ since $2 = 1 + 1$. If it holds for n, then

$$2S_{n+1} = 2S_n + 2a_{n+1} = a_{n+2} + 2a_{n+1} + a_n = a_{n+3} + a_{n+1}$$

by the recurrence, completing the induction.

Therefore $S_{28} = \frac{a_{30}+a_{28}}{2} = \frac{20603361+6090307}{2} = 13346834$, whose remainder upon division by 1000 is 834.

12. Equilateral $\triangle ABC$ is inscribed in a circle of radius 2. Extend $\overline{AB}$ through B to point D so that $AD = 13$, and extend $\overline{AC}$ through C to point E so that $AE = 11$. Through D, draw a line ℓ_1 parallel to $\overline{AE}$, and through E, draw a line ℓ_2 parallel to $\overline{AD}$. Let F be the intersection of ℓ_1 and ℓ_2. Let G be the point on the circle that is collinear with A and F and distinct from A. Given that the area of $\triangle CBG$ can be expressed in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime, find $p + q + r$.

Solution:

By construction $ADFE$ is a parallelogram with $AD = 13, DF = AE = 11$, and $\angle ADF = 180^\circ - \angle DAE = 120^\circ$. Hence $[ADF] = \frac{1}{2} \cdot 13 \cdot 11 \sin 120^\circ = \frac{143\sqrt{3}}{4}$, and by the law of cosines,

$$AF^2 = 13^2 + 11^2 - 2 \cdot 13 \cdot 11 \cos 120^\circ = 169 + 121 + 143 = 433.$$

Since G lies on the circle, inscribed angles give $\angle GCB = \angle GAB = \angle FAD$ (both subtend arc GB) and $\angle CBG = \angle CAG$ (both subtend arc CG); and $\angle CAG = \angle AFD$ because $\overline{AE} \parallel \overline{DF}$. So $\triangle CBG \sim \triangle AFD$ with ratio $\frac{CB}{AF}$. The side of an equilateral triangle inscribed in a circle of radius 2 is $BC = 2\sqrt{3}$.

Therefore

$$[CBG] = \left(\frac{2\sqrt{3}}{\sqrt{433}}\right)^2 \cdot \frac{143\sqrt{3}}{4} = \frac{12}{433} \cdot \frac{143\sqrt{3}}{4} = \frac{429\sqrt{3}}{433},$$

and $p + q + r = 429 + 3 + 433 = 865.$

13. How many integers N less than 1000 can be written as the sum of j consecutive positive odd integers for exactly 5 values of $j \geq 1$?

Solution:

The sum of the $(k+1)$th through mth positive odd integers is $m^2 - k^2 = (m-k)(m+k)$. Writing $a = m - k$ and $b = m + k$, the representations of N correspond exactly to the factorizations $N = ab$ with $a \leq b$ and a, b of the same parity (then $m = \frac{a+b}{2}, k = \frac{b-a}{2}$). So we need N to have exactly 5 such factorizations.

If N is odd, every divisor pair works, so N needs 9 or 10 divisors, i.e. $N = p^8, p^9, p^2q^2$, or pq^4 with p, q distinct odd primes. Below 1000, p^8 and p^9 are impossible, p^2q^2 gives 225 and 441, and pq^4 gives $3^4 \cdot 5, 3^4 \cdot 7, 3^4 \cdot 11$: five odd values.

If N is even, both factors must be even, so $N = 4M$ and the factorizations correspond to divisor pairs of M, with no parity restriction; we need $M < 250$ with 9 or 10 divisors. With 9 divisors: $36, 100, 196, 225$. With 10 divisors (pq^4): $3 \cdot 2^4, 5 \cdot 2^4$, $7 \cdot 2^4, 11 \cdot 2^4, 13 \cdot 2^4, 2 \cdot 3^4$. That is $4 + 6 = 10$ even values, for a total of $5 + 10 = 15$.

14. Let S_n be the sum of the reciprocals of the nonzero digits of the integers from 1 to 10^n, inclusive. Find the smallest positive integer n for which S_n is an integer.

Solution:

Write the integers from 0 to $10^n - 1$ as n-digit strings with leading zeros. Each of the n digit positions takes each digit value equally often, so each nonzero digit appears $n \cdot 10^{n-1}$ times. Adding the digit 1 of 10^n itself,

$$S_n = 1 + n \cdot 10^{n-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{9}\right) = 1 + \frac{7129}{2520}\, n \cdot 10^{n-1}.$$

Since $\gcd(7129, 2520) = 1$, the sum is an integer exactly when $2520 \mid n \cdot 10^{n-1}$. Now $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$, and for $n \geq 4$ the factor 10^{n-1} supplies $2^3 \cdot 5$, leaving the condition $63 \mid n$ (a power of 10 has no factors of 3 or 7). For $n = 1, 2, 3$ the products $1, 20, 300$ are not multiples of 2520.

The smallest solution is therefore $n = 63$.

15. Given that x, y, and z are real numbers that satisfy

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}},$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}},$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}},$$

and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers, and n is not divisible by the square of any prime, find $m + n$.

Solution:

Each radical $\sqrt{y^2 - \frac{1}{16}}$ is the leg of a right triangle with hypotenuse y and other leg $\frac{1}{4}$. So the first equation says: in a triangle XYZ with $x = YZ, y = ZX, z = XY$, the altitude from X has length $\frac{1}{4}$, and its foot splits YZ into the two radical lengths. The other equations say the altitudes to sides y and z are $\frac{1}{5}$ and $\frac{1}{6}$.

If K is the area of this triangle, then $K = \frac{1}{2} \cdot x \cdot \frac{1}{4}$ gives $x = 8K$, and likewise $y = 10K$ and $z = 12K$. These are proportional to $8, 10, 12$, and $8^2 + 10^2 > 12^2$, so the triangle is acute and the altitude feet do land inside the sides. Heron's formula with $s = 15K$ gives

$$K^2 = 15K \cdot 7K \cdot 5K \cdot 3K = 1575K^4,$$

so $K^2 = \frac{1}{1575}$ and $K = \frac{1}{15\sqrt{7}}$.

Then $x + y + z = 30K = \frac{2}{\sqrt{7}}$, so $m + n = 2 + 7 = 9$.

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