# 2005 AIME II Solutions

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1. A game uses a deck of $n$ different cards, where $n$ is an integer and $n \geq 6$. The number of possible sets of $6$ cards that can be drawn from the deck is $6$ times the number of possible sets of $3$ cards that can be drawn. Find $n$.

## Solution:

The condition says $\binom{n}{6} = 6\binom{n}{3}$. Dividing the binomial coefficients,

$$\frac{\binom{n}{6}}{\binom{n}{3}} = \frac{(n-3)(n-4)(n-5)}{6 \cdot 5 \cdot 4} = 6,$$

so $(n-3)(n-4)(n-5) = 720 = 10 \cdot 9 \cdot 8$.

Since the product $(n-3)(n-4)(n-5)$ is increasing in $n$, the only solution is $n - 3 = 10$, that is, $n = 13$.

**2.** A hotel packed a breakfast for each of three guests. Each breakfast should have consisted of three types of rolls, one each of nut, cheese, and fruit rolls. The preparer wrapped each of the nine rolls, and, once they were wrapped, the rolls were indistinguishable from one another. She then randomly put three rolls in a bag for each of the guests. Given that the probability that each guest got one roll of each type is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, find $m+n$.

## Solution:

Fill the first guest's bag one roll at a time. The first roll can be anything; the second must avoid the $2$ remaining rolls of the first roll's type, succeeding with probability $\frac{6}{8}$; and the third must be one of the $3$ rolls of the missing type among the remaining $7$. So the first bag has one roll of each type with probability $\frac{6}{8} \cdot \frac{3}{7} = \frac{9}{28}$.

Given that, six rolls remain, two of each type, and the same argument gives $\frac{4}{5} \cdot \frac{2}{4} = \frac{2}{5}$ for the second bag. The third bag is then automatically one of each type. The probability is $\frac{9}{28} \cdot \frac{2}{5} = \frac{9}{70}$, so $m+n = 9+70 = 79$.

**3.** An infinite geometric series has sum $2005$. A new series, obtained by squaring each term of the original series, has sum $10$ times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

## Solution:

Let the original series have first term $a$ and ratio $r$, so $\frac{a}{1-r} = 2005$. The squared series is geometric with first term $a^2$ and ratio $r^2$, so

$$\frac{a^2}{1-r^2} = \frac{a}{1-r} \cdot \frac{a}{1+r} = 2005 \cdot \frac{a}{1+r} = 10 \cdot 2005,$$

which gives $\frac{a}{1+r} = 10$.

Dividing the two equations, $\frac{1+r}{1-r} = \frac{2005}{10}$, so $2(1+r) = 401(1-r)$, giving $403r = 399$ and $r = \frac{399}{403}$. Since $399 = 3 \cdot 7 \cdot 19$ and $403 = 13 \cdot 31$, the fraction is in lowest terms, and $m+n = 399+403 = 802$.

**4.** Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$.

## Solution:

From the factorizations $10^{10} = 2^{10}5^{10}$, $15^7 = 3^7 5^7$, and $18^{11} = 2^{11}3^{22}$, the divisor counts are $11 \cdot 11 = 121$, $8 \cdot 8 = 64$, and $12 \cdot 23 = 276$.

The divisors common to two of the numbers are exactly the divisors of their gcd: $\gcd(10^{10}, 15^7) = 5^7$ has $8$ divisors, $\gcd(10^{10}, 18^{11}) = 2^{10}$ has $11$, and $\gcd(15^7, 18^{11}) = 3^7$ has $8$. Only $1$ divides all three numbers.

By inclusion-exclusion, the count is $121 + 64 + 276 - 8 - 11 - 8 + 1 = 435$.

5. Determine the number of ordered pairs $(a, b)$ of integers such that $\log_a b + 6 \log_b a = 5, 2 \leq a \leq 2005$, and $2 \leq b \leq 2005$.

## Solution:

Let $x = \log_a b$. Since $\log_b a = \frac{1}{x}$, the equation becomes $x + \frac{6}{x} = 5$, i.e. $x^2 - 5x + 6 = 0$, so $x = 2$ or $x = 3$. That means $b = a^2$ or $b = a^3$.

For $b = a^2 \leq 2005$ we need $2 \leq a \leq 44$ (since $44^2 = 1936$ and $45^2 = 2025$), giving $43$ pairs. For $b = a^3 \leq 2005$ we need $2 \leq a \leq 12$ (since $12^3 = 1728$ and $13^3 = 2197$), giving $11$ pairs.

In total there are $43 + 11 = 54$ ordered pairs.

6. The cards in a stack of $2n$ cards are numbered consecutively from $1$ through $2n$ from top to bottom. The top $n$ cards are removed, kept in order, and form pile $A$. The remaining cards form pile $B$. The cards are now restacked into a single stack by taking cards alternately from the tops of pile $B$ and pile $A$, respectively. In this process, card number $(n+1)$ is the bottom card of the new stack, card number $1$ is on top of this card, and so on, until piles $A$ and $B$ are exhausted. If, after the restacking process, at least one card from each pile occupies the same position that it occupied in the original stack, the stack is called *magical*. For example, eight cards form a magical stack because cards number $3$ and number $6$ retain their original positions. Find the number of cards in the magical stack in which card number $131$ retains its original position.

## Solution:

The new stack, read from the bottom up, is $n+1,\ 1,\ n+2,\ 2,\ \ldots,\ 2n,\ n$. So pile $B$'s cards occupy the even positions from the top in reverse order, and pile $A$'s cards occupy the odd positions in reverse order: a card at original position $i \le n$ (pile $A$) moves to position $2(n-i)+1$, while a card at position $i > n$ (pile $B$) moves to position $2(2n-i)+2$.

Since $131$ is odd, card $131$ can keep its position only if it comes from pile $A$, so $131 = 2(n-131)+1$, which gives $n = 196$. Indeed $131 \le 196$, and the stack is magical because card $262$ from pile $B$ also stays fixed: $2(2n-262)+2 = 262$. The stack has $2n = 392$ cards.

**7.** Let

$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Find $(x+1)^{48}$.

## Solution:

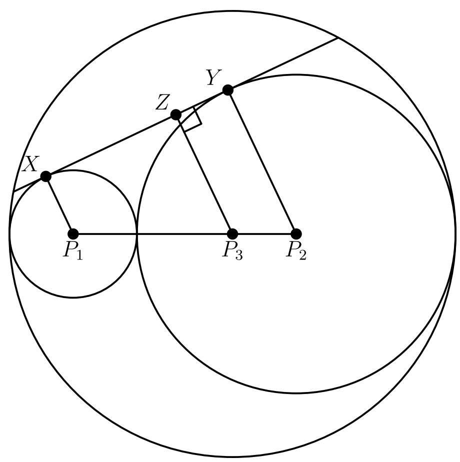
Let $y = \sqrt[16]{5}$. Multiplying the numerator and denominator by $y-1$ telescopes the denominator by repeated difference of squares:

$$x = \frac{4(y-1)}{(y^8+1)(y^4+1)(y^2+1)(y+1)(y-1)} = \frac{4(y-1)}{y^{16}-1} = \frac{4(y-1)}{4} = y-1.$$

Hence $x+1 = y = 5^{1/16}$, and $(x+1)^{48} = 5^{48/16} = 5^3 = 125$.

**8.** Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are externally tangent, and they are both internally tangent to circle $\mathcal{C}_3$. The radii of $\mathcal{C}_1$ and $\mathcal{C}_2$ are $4$ and $10$, respectively, and the centers of the three circles are all collinear. A chord of $\mathcal{C}_3$ is also a common external tangent of $\mathcal{C}_1$ and $\mathcal{C}_2$. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$, where $m, n,$ and $p$ are positive integers, $m$ and $p$ are relatively prime, and $n$ is not divisible by the square of any prime, find $m+n+p$.

## Solution:




Let $P_1, P_2, P_3$ be the centers of the circles and $R$ the radius of $\mathcal{C}_3$. External tangency gives $P_1P_2 = 4+10 = 14$, and internal tangency gives $P_3P_1 = R-4$ and $P_3P_2 = R-10$. Since the centers are collinear, $(R-4)+(R-10) = 14$, so $R = 14$ and $P_3$ lies on $\overline{P_1P_2}$ with $P_3P_1 = 10$ and $P_3P_2 = 4$.

Drop perpendiculars $P_1X, P_2Y, P_3Z$ to the chord, so $P_1X = 4$, $P_2Y = 10$, and $Z$ is the midpoint of the chord. The distance from a point moving along line $P_1P_2$ to the tangent line changes linearly, and $P_3$ is $\frac{10}{14}$ of the way from $P_1$ to $P_2$, so

$$P_3Z = 4 + \frac{10}{14}(10-4) = \frac{58}{7}.$$

The half-chord is $\sqrt{14^2 - \left(\frac{58}{7}\right)^2} = \frac{\sqrt{9604-3364}}{7} = \frac{4\sqrt{390}}{7}$, so the chord has length $\frac{8\sqrt{390}}{7}$. Since $390 = 2\cdot 3\cdot 5\cdot 13$ is squarefree and $\gcd(8,7) = 1$, the answer is $m+n+p = 8+390+7 = 405$.

**9.** For how many positive integers $n$ less than or equal to $1000$ is

$$(\sin t + i \cos t)^n = \sin nt + i \cos nt$$

true for all real $t$?

## Solution:

Since $\sin t + i\cos t = i(\cos t - i\sin t)$ and $\sin nt + i\cos nt = i(\cos nt - i\sin nt)$, de Moivre's theorem (applied to angle $-t$) gives

$$(\sin t + i\cos t)^n = i^n(\cos t - i\sin t)^n = i^n(\cos nt - i\sin nt).$$

So the equation holds for all real $t$ exactly when $i^n = i$, that is, when $n \equiv 1 \pmod 4$. The values $n = 1, 5, 9, \ldots, 997$ give exactly $250$ positive integers up to $1000$.

**10.** Given that $\mathcal{O}$ is a regular octahedron, that $\mathcal{C}$ is the cube whose vertices are the centers of the faces of $\mathcal{O}$, and that the ratio of the volume of $\mathcal{O}$ to that of $\mathcal{C}$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, find $m+n$.

## Solution:

Place the octahedron's vertices at $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$. It is two square pyramids glued along the square with vertices $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$, which has area $2$, and each pyramid has height $1$, so

$$V_{\mathcal{O}} = 2 \cdot \frac{1}{3} \cdot 2 \cdot 1 = \frac{4}{3}.$$

Each face centroid is the average of that face's three vertices, e.g. $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, so the cube has vertices $\left(\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}\right)$. Its edge is $\frac{2}{3}$ and its volume is $\frac{8}{27}$.

The ratio is $\frac{4/3}{8/27} = \frac{9}{2}$, so $m+n = 9+2 = 11$.

**11.** Let $m$ be a positive integer, and let $a_0, a_1, \ldots, a_m$ be a sequence of real numbers such that $a_0 = 37, a_1 = 72, a_m = 0,$ and

$$a_{k+1} = a_{k-1} - \frac{3}{a_k}$$

for $k = 1, 2, \ldots, m-1.$ Find $m.$

## Solution:

Multiplying the recurrence by $a_k$ gives $a_{k+1}a_k = a_k a_{k-1} - 3,$ so the products $b_k = a_k a_{k-1}$ form an arithmetic sequence with common difference $-3.$ Since $b_1 = 72 \cdot 37 = 2664 = 3 \cdot 888,$ we get

$$b_k = 2664 - 3(k-1) = 3(889 - k).$$

Thus $b_k > 0$ for $k \le 888,$ so no term before $a_{889}$ can vanish (and the recurrence never divides by zero), while $b_{889} = a_{889}a_{888} = 0$ with $a_{888} \neq 0.$ Hence $a_{889} = 0,$ and $m = 889.$

**12.** Square $ABCD$ has center $O$, $AB = 900$, $E$ and $F$ are on $\overline{AB}$ with $AE < BF$ and $E$ between $A$ and $F$, $m\angle EOF = 45^\circ$, and $EF = 400$. Given that $BF = p + q\sqrt{r}$, where $p, q,$ and $r$ are positive integers and $r$ is not divisible by the square of any prime, find $p + q + r$.

## Solution:

Let $G$ be the midpoint of $\overline{AB}$, so $OG \perp AB$ and $OG = 450$. With $\alpha = \angle EOG$ and $\beta = \angle FOG$ on either side of ray $OG$, we have $EG = 450 \tan\alpha$, $FG = 450 \tan\beta$, and $\alpha + \beta = 45^\circ$. From $EG + FG = EF = 400$, we get $\tan\alpha + \tan\beta = \frac{8}{9}$.

The tangent addition formula gives

$$1 = \tan 45^\circ = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

so $\tan\alpha\tan\beta = 1 - \frac{8}{9} = \frac{1}{9}$. Hence $\tan\alpha$ and $\tan\beta$ are the roots of $9t^2 - 8t + 1 = 0$, namely $\frac{4\pm\sqrt{7}}{9}$.

Since $AE = 450 - EG$ and $BF = 450 - FG$, the condition $AE < BF$ means $EG > FG$, so $\tan\beta = \frac{4-\sqrt{7}}{9}$. Then $BF = 450 - 450 \cdot \frac{4-\sqrt{7}}{9} = 250 + 50\sqrt{7}$, and $p + q + r = 250 + 50 + 7 = 307$.

**13.** Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that the equation $P(n) = n + 3$ has two distinct integer solutions $n_1$ and $n_2$, find the product $n_1 \cdot n_2$.

## Solution:

Let $S(x) = P(x) - x - 3$, so $S(17) = S(24) = -10$. Since $S(x) + 10$ has integer coefficients and vanishes at $17$ and $24$,

$$S(x) = -10 + (x - 17)(x - 24)Q(x)$$

for some polynomial $Q$ with integer coefficients.

If $P(n) = n + 3$ for an integer $n$, then $(n - 17)(n - 24)Q(n) = 10$, so $(n - 17)(n - 24)$ divides $10$. The factors $n - 17$ and $n - 24$ are integers differing by $7$ whose product divides $10$, so they are $\{2, -5\}$ or $\{5, -2\}$, giving $n = 19$ and $n = 22$. Both occur, for example, for $P(x) = x - 7 - (x - 17)(x - 24)$.

Hence $n_1 \cdot n_2 = 19 \cdot 22 = 418$.

**14.** In triangle $ABC, AB = 13, BC = 15$, and $CA = 14$. Point $D$ is on $\overline{BC}$ with $CD = 6$. Point $E$ is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers, find $q$.

## Solution:

A cevian $AD$ splits the opposite side in the ratio

$$\frac{BD}{DC} = \frac{[ABD]}{[ACD]} = \frac{\frac{1}{2}AB \cdot AD \sin \angle BAD}{\frac{1}{2}AC \cdot AD \sin \angle CAD} = \frac{AB \sin \angle BAD}{AC \sin \angle CAD},$$

and similarly $\frac{BE}{EC} = \frac{AB \sin \angle BAE}{AC \sin \angle CAE}$.

Since $\angle BAE = \angle CAD$, we also have $\angle BAD = \angle CAE$ (each is that common angle plus $\angle EAD$), so multiplying the two ratios cancels all the sines: $\frac{BD}{DC} \cdot \frac{BE}{EC} = \frac{AB^2}{AC^2}$. With $BD = 9, DC = 6, AB = 13$, and $AC = 14$, this gives

$$\frac{BE}{EC} = \frac{13^2}{14^2} \cdot \frac{6}{9} = \frac{169}{294}.$$

Hence $BE = 15 \cdot \frac{169}{169+294} = \frac{2535}{463}$. Since $463$ is prime and does not divide $2535 = 3 \cdot 5 \cdot 13^2$, the fraction is in lowest terms, and $q = 463$.

**15.** Let $\omega_1$ and $\omega_2$ denote the circles $x^2+y^2+10x-24y-87=0$ and $x^2+y^2-10x-24y+153=0$, respectively. Let $m$ be the smallest positive value of $a$ for which the line $y=ax$ contains the center of a circle that is internally tangent to $\omega_1$ and externally tangent to $\omega_2$. Given that $m^2=\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers, find $p+q$.

## Solution:

Completing the square gives $\omega_1:\ (x+5)^2+(y-12)^2=256$ and $\omega_2:\ (x-5)^2+(y-12)^2=16$, with centers $F_1=(-5,12)$ and $F_2=(5,12)$ and radii $16$ and $4$. If a circle with center $P$ and radius $r$ is internally tangent to $\omega_1$ and externally tangent to $\omega_2$, then $PF_1=16-r$ and $PF_2=4+r$, so $PF_1+PF_2=20$.

Thus $P$ lies on the ellipse with foci $F_1$ and $F_2$ and major axis $20$ : the semimajor axis is $10$, the center-to-focus distance is $5$, so the semiminor axis squared is $100-25=75$, giving

$$\frac{x^2}{100}+\frac{(y-12)^2}{75}=1,$$

i.e. $3x^2+4y^2-96y+576=300$. Substituting $y=ax$ yields $(3+4a^2)x^2-96ax+276=0$.

The line $y=ax$ contains such a center exactly when this quadratic has a real root, i.e. $(96a)^2-4\cdot 276\,(3+4a^2)\geq 0$, which simplifies to $4800a^2\geq 3312$, so $a^2\geq\frac{69}{100}$. The smallest positive such $a$ has $m^2=\frac{69}{100}$, and $p+q=69+100=169$.

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