2005 AIME I Solutions

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1. Six congruent circles form a ring with each circle externally tangent to the two circles adjacent to it. All six circles are internally tangent to a circle $\mathcal{C}$ with radius 30. Let K be the area of the region inside $\mathcal{C}$ and outside all of the six circles in the ring. Find $\lfloor K \rfloor$. (The notation $\lfloor K \rfloor$ denotes the greatest integer that is less than or equal to K.)

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Solution:

Let r be the common radius of the six circles. Adjacent circles are externally tangent, so their centers are $2r$ apart, and the six centers form a regular hexagon with side $2r$. Since a regular hexagon's circumradius equals its side length, each center is at distance $2r$ from the center O of $\mathcal{C}$. Internal tangency to $\mathcal{C}$ means the distance from O to each small center plus r equals 30, so $3r = 30$ and $r = 10$.

Therefore

$$K = \pi \left(30^2 - 6 \cdot 10^2\right) = 300\pi \approx 942.48,$$

and $\lfloor K \rfloor = 942$.

2. For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, \ldots$. For how many values of k does S_k contain the term 2005?

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Solution:

The nth term of S_k is $1 + (n-1)k$, so 2005 is a term exactly when $(n-1)k = 2004$ for some positive integer n, that is, exactly when k divides 2004. Every divisor works, since $n - 1 = \frac{2004}{k}$ is then a positive integer.

Since $2004 = 2^2 \cdot 3 \cdot 167$, the number of divisors is $(2+1)(1+1)(1+1) = 12$.

3. How many positive integers have exactly three proper divisors, each of which is less than 50? (A *proper divisor* of a positive integer n is a positive integer divisor of n other than n itself.)

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Solution:

An integer with exactly three proper divisors has exactly four divisors in total, so it is either $n = pq$ with p and q distinct primes (proper divisors $1, p, q$) or $n = p^3$ with p prime (proper divisors $1, p, p^2$).

In the first case we need p and q both less than 50. There are 15 primes below 50, giving $\binom{15}{2} = 105$ such numbers. In the second case we need $p^2 < 50$, which holds for $p = 2, 3, 5, 7$, giving 4 more.

The total is $105 + 4 = 109$.

4. The director of a marching band wishes to place the members into a formation that includes all of them and has no unfilled positions. If they are arranged in a square formation, there are 5 members left over. The director finds that if they are arranged in a rectangular formation with 7 more rows than columns, the desired result can be obtained. Find the maximum number of members this band can have.

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Solution:

Let the band have n members, with $n = s^2 + 5$ for the square formation and $n = x(x + 7)$ for the rectangular formation with x columns. Multiplying $x^2 + 7x = s^2 + 5$ by 4 and completing the square gives

$$(2x + 7)^2 - (2s)^2 = 69,$$

so $(2x + 7 - 2s)(2x + 7 + 2s) = 69$.

Writing $69 = 1 \cdot 69 = 3 \cdot 23$ with the larger factor second: from $1 \cdot 69$ we get $2x + 7 = 35$ and $2s = 34$, so $x = 14, s = 17$, and $n = 17^2 + 5 = 294$. From $3 \cdot 23$ we get $2x + 7 = 13$ and $2s = 10$, so $x = 3, s = 5$, and $n = 30$.

The maximum is 294, achieved by a 21×14 rectangle.

5. Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of a face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

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Solution:

Choose the coin orientations and the gold/silver positions independently. Record the orientations from bottom to top as a string of U (engraved face up) and D (engraved face down). Two adjacent coins are face to face exactly when the lower coin's engraved side faces up and the upper coin's engraved side faces down — that is, exactly when a U is immediately followed by a D.

A string of U's and D's avoids the pattern UD exactly when every D precedes every U, so the string is $\mathrm{D}^i\mathrm{U}^{8-i}$ for some $i = 0, 1, \ldots, 8$: there are 9 allowable orientation strings. Independently, the gold coins occupy 4 of the 8 positions in $\binom{8}{4} = 70$ ways.

The total is $9 \cdot 70 = 630$.

6. Let P be the product of the nonreal roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$. Find $\lfloor P \rfloor$. (The notation $\lfloor P \rfloor$ denotes the greatest integer that is less than or equal to P.)

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Solution:

Adding 1 to both sides turns the left side into a perfect fourth power:

$$(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1 = 2006.$$

So $x - 1$ is a fourth root of 2006 : the four roots are $x = 1 \pm \sqrt[4]{2006}$ (real) and $x = 1 \pm i\sqrt[4]{2006}$ (nonreal).

The product of the conjugate pair of nonreal roots is

$$\left(1 + i\sqrt[4]{2006}\right)\left(1 - i\sqrt[4]{2006}\right) = 1 + \sqrt{2006}.$$

Since $44^2 = 1936 < 2006 < 2025 = 45^2$, we have $45 < P < 46$, so $\lfloor P \rfloor = 45$.

7. In quadrilateral $ABCD, BC = 8, CD = 12, AD = 10$, and $m\angle A = m\angle B = 60^\circ$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p + q$.

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Solution:

Extend rays AD and BC until they meet at P. Triangle ABP has 60° angles at A and B, so it is equilateral: $PA = PB = AB$. Writing $x = AB$, we get $PD = PA - AD = x - 10$ and $PC = PB - BC = x - 8$.

The Law of Cosines in triangle PDC, with $\angle P = 60^\circ$ and $DC = 12$, gives

$$144 = (x-10)^2 + (x-8)^2 - (x-10)(x-8) = x^2 - 18x + 84,$$

so $x^2 - 18x - 60 = 0$ and $x = 9 + \sqrt{81 + 60} = 9 + \sqrt{141}$.

Thus $p + q = 9 + 141 = 150$.

8. The equation

$$2^{333x-2}+2^{111x+2}=2^{222x+1}+1$$

has three real roots. Given that their sum is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

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Solution:

Let $y=2^{111x}$. Then $2^{333x-2}=\frac{y^3}{4}$, $2^{111x+2}=4y$, and $2^{222x+1}=2y^2$, so the equation becomes $\frac{y^3}{4}+4y=2y^2+1$, that is,

$$y^3-8y^2+16y-4=0.$$

Since the three roots x_1,x_2,x_3 are real and $y=2^{111x}$ is strictly increasing, they correspond to three positive real roots y_1,y_2,y_3 of the cubic.

Each $x_i=\frac{1}{111}\log_2 y_i$, so

$$x_1+x_2+x_3=\frac{1}{111}\log_2(y_1y_2y_3)=\frac{1}{111}\log_2 4=\frac{2}{111},$$

using Vieta's formulas for the product of the roots. Then $m+n=2+111=113.$

9. Twenty-seven unit cubes are each painted orange on a set of four faces so that the two unpainted faces share an edge. The 27 cubes are then randomly arranged to form a $3\times3\times3$ cube. Given that the probability that the entire surface of the larger cube is orange is $\frac{p^a}{q^br^c}$, where $p,q,$ and r are distinct primes and $a,b,$ and c are positive integers, find $a+b+c+p+q+r$.

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Solution:

Each unit cube has one "bad edge": the edge shared by its two unpainted faces. The larger cube's surface is entirely orange exactly when every unit cube's bad edge touches no visible face. A uniformly random orientation places the bad edge uniformly among the cube's 12 edge positions, so for each unit cube we count the edge positions both of whose faces are hidden.

A corner cube shows 3 faces meeting at a vertex; the safe edges are those of the 3 hidden faces meeting at the opposite vertex, so the probability is $\frac{3}{12}=\frac{1}{4}$. An edge cube shows 2 adjacent faces, which touch $4+4-1=7$ edges, leaving 5 safe: probability $\frac{5}{12}$. A face-center cube shows 1 face touching 4 edges, leaving 8 safe: probability $\frac{8}{12}=\frac{2}{3}$. The center cube is always fine.

With 8 corner, 12 edge, and 6 face-center cubes, the probability is

$$\left(\frac{1}{4}\right)^8\left(\frac{5}{12}\right)^{12}\left(\frac{2}{3}\right)^6=\frac{5^{12}}{2^{34}\cdot3^{18}},$$

so $a+b+c+p+q+r=12+34+18+5+2+3=74.$

10. Triangle ABC lies in the Cartesian plane and has area 70. The coordinates of B and C are $(12, 19)$ and $(23, 20)$, respectively, and the coordinates of A are (p, q). The line containing the median to side $\overline{BC}$ has slope -5. Find the largest possible value of $p + q$.

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Solution:

The median to $\overline{BC}$ passes through the midpoint $M = \left(\frac{35}{2}, \frac{39}{2}\right)$ of $\overline{BC}$. The line through M with slope -5 is $y = -5x + 107$, and A lies on this line, so $A = (p, -5p + 107)$ and $q = -5p + 107$.

By the shoelace formula with $B = (12, 19)$ and $C = (23, 20)$,

$$[ABC] = \frac{1}{2}\left|-p + 12\big(20 - q\big) + 23\big(q - 19\big)\right| = \frac{1}{2}\left|980 - 56p\right| = 70,$$

so $|56p - 980| = 140$, giving $p = 15$ or $p = 20$.

Since $p + q = p + (-5p + 107) = 107 - 4p$, the smaller value $p = 15$ gives the larger sum $107 - 60 = 47$.

11. A semicircle with diameter d is contained in a square whose sides have length 8. Given that the maximum value of d is $m - \sqrt{n}$, where m and n are integers, find $m + n$.

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Solution:

Scale to a semicircle of radius 1 and ask for the smallest square containing it when its diameter makes angle θ with one pair of sides, where $0 \le \theta \le 90^\circ$. Squeeze the semicircle between two pairs of parallel lines in the square's two side directions: in each direction one line of the pair is tangent to the arc and the other passes through an endpoint of the diameter, and the distances between the pairs are $1 + \cos\theta$ and $1 + \sin\theta$. So the smallest enclosing square in that orientation has side $\max\{1 + \cos\theta,\ 1 + \sin\theta\}$, which is minimized when $\theta = 45^\circ$, giving side $1 + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{2}$.

Scaling this optimal configuration so the square has side 8, the radius becomes $r = \frac{8}{(2+\sqrt{2})/2} = \frac{16}{2+\sqrt{2}} = 8\left(2 - \sqrt{2}\right)$, so

$$d = 2r = 16\left(2 - \sqrt{2}\right) = 32 - 16\sqrt{2} = 32 - \sqrt{512}.$$

Thus $m + n = 32 + 512 = 544.$

12. For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by

$$S(n) = \tau(1) + \tau(2) + \cdots + \tau(n).$$

Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

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Solution:

Divisors of n pair up as d and $\frac{n}{d}$, so $\tau(n)$ is odd exactly when n is a perfect square. Hence $S(n)$ changes parity exactly at the squares, which means $S(n)$ is odd exactly when the number of squares up to n, namely $\lfloor \sqrt{n} \rfloor$, is odd.

For each k, there are $2k + 1$ integers n with $\lfloor \sqrt{n} \rfloor = k$, namely $k^2 \leq n \leq k^2 + 2k$. Since $44^2 = 1936 \leq 2005 < 2025 = 45^2$, the odd values $k = 1, 3, \ldots, 43$ all have their full blocks within range, so

$$a = \sum_{k \text{ odd}, \, k \leq 43} (2k+1) = 2(1 + 3 + \cdots + 43) + 22 = 2 \cdot 484 + 22 = 990.$$

Then $b = 2005 - 990 = 1015$, and $|a - b| = 25$.

13. A particle moves in the Cartesian plane from one lattice point to another according to the following rules:

- From any lattice point (a, b), the particle may move only to $(a+1, b)$, $(a, b+1)$, or $(a+1, b+1)$.
- There are no right angle turns in the particle's path. That is, the sequence of points visited contains neither a subsequence of the form $(a, b), (a+1, b), (a+1, b+1)$ nor a subsequence of the form $(a, b), (a, b+1), (a+1, b+1)$.

How many different paths can the particle take from $(0, 0)$ to $(5, 5)$?

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Solution:

The forbidden right-angle turns say exactly that a rightward step may never immediately follow an upward step, and vice versa; a diagonal step may follow or precede anything. So at each lattice point (x, y) track three counts $D(x, y), R(x, y), U(x, y)$: the numbers of legal paths from $(0, 0)$ arriving there by a diagonal, rightward, or upward step. The rules give

$$D(x,y) = D + R + U \text{ at } (x-1, y-1), \quad R(x,y) = D(x-1,y) + R(x-1,y), \quad U(x,y) = D(x,y-1) + U(x,y-1).$$

Starting from the single empty path at $(0, 0)$ (which may begin with any step), fill in the grid up to $(5, 5)$. Along the axes only all-rightward or all-upward paths survive, and the interior builds up quickly; at $(5, 5)$ the three counts come out to $27, 28$, and 28.

The total number of paths is $27 + 28 + 28 = 83$.

14. Consider the points $A(0,12), B(10,9), C(8,0)$, and $D(-4,7)$. There is a unique square $\mathcal{S}$ such that each of the four points is on a different side of $\mathcal{S}$. Let K be the area of $\mathcal{S}$. Find the remainder when $10K$ is divided by 1000.

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Solution:

Since segments $\overline{AC}$ and $\overline{BD}$ cross, A and C lie on opposite sides of the square, as do B and D. Let m be the slope of the side through B, so that side lies on $mx - y + 9 - 10m = 0$, and the perpendicular side through C lies on $x + my - 8 = 0$. The side length of the square equals both the distance between the parallel sides through B and D and the distance between the sides through A and C :

$$\frac{|-4m - 7 + 9 - 10m|}{\sqrt{m^2+1}} = \frac{|12m - 8|}{\sqrt{m^2+1}},$$

so $|2 - 14m| = |12m - 8|$, giving $m = \frac{5}{13}$ or $m = -3$.

For $m = \frac{5}{13}$, the points A and C fall on opposite sides of the line through B, which is impossible if that line contains a side of the square, so $m = -3$. Then the side length is $\frac{|12(-3)-8|}{\sqrt{9+1}} = \frac{44}{\sqrt{10}}$, so

$$K = \frac{44^2}{10} = \frac{1936}{10} \qquad \text{and} \qquad 10K = 1936.$$

The remainder when 1936 is divided by 1000 is 936.

15. In $\triangle ABC$, $AB = 20$. The incircle of the triangle divides the median containing C into three segments of equal length. Given that the area of $\triangle ABC$ is $m\sqrt{n}$, where m and n are integers and n is not divisible by the square of any prime, find $m + n$.

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Solution:

Let M be the midpoint of $\overline{AB}$, and let the incircle cut median $\overline{CM}$ at S and N, with $CS = SN = NM = \frac{1}{3}CM$. Let the incircle touch $\overline{AB}$ at T and $\overline{AC}$ at R. By Power of a Point,

$$MT^2 = MN \cdot MS = \frac{CM}{3} \cdot \frac{2\,CM}{3} = \frac{2}{9}CM^2, \qquad CR^2 = CS \cdot CN = \frac{2}{9}CM^2,$$

so $MT = CR$. Since $AR = AT$ (tangents from A), we get $AC = AR + RC = AT + TM = AM = 10$.

Write $a = BC$ and $s = \frac{20+a+10}{2} = 15 + \frac{a}{2}$. The standard tangent length gives $AT = s - a$, so $MT = AM - AT = 10 - \left(15 - \frac{a}{2}\right) = \frac{a-10}{2}$, while the median length formula gives $CM^2 = \frac{2\cdot 10^2 + 2a^2 - 20^2}{4} = \frac{a^2-100}{2}$. Substituting into $MT^2 = \frac{2}{9}CM^2$:

$$\frac{(a-10)^2}{4} = \frac{a^2-100}{9} \implies 9(a-10) = 4(a+10) \implies a = 26.$$

Then the sides are $20, 26, 10$ with $s = 28$, and Heron's formula gives

$$[ABC] = \sqrt{28 \cdot 8 \cdot 2 \cdot 18} = \sqrt{8064} = 24\sqrt{14},$$

so $m + n = 24 + 14 = 38$.

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