2004 AIME II Solutions

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1. A chord of a circle is perpendicular to a radius at the midpoint of the radius. The ratio of the area of the larger of the two regions into which the chord divides the circle to the smaller can be expressed in the form $\frac{a\pi+b\sqrt{c}}{d\pi-e\sqrt{f}}$, where $a, b, c, d, e,$ and f are positive integers, a and e are relatively prime, and neither c nor f is divisible by the square of any prime. Find the remainder when the product $a \cdot b \cdot c \cdot d \cdot e \cdot f$ is divided by 1000.

Solution:

Scale so the radius is 2. The chord lies at distance 1 from the center, so each radius to an endpoint of the chord makes a 60° angle with the bisected radius, and the two endpoint radii form a central angle of 120°. The isosceles triangle they cut off has area $\frac{1}{2} \cdot 2 \cdot 2 \sin 120^\circ = \sqrt{3}$, and the whole disk has area 4π.

The smaller region is the 120° sector minus the triangle, $\frac{4\pi}{3} - \sqrt{3}$, and the larger region is the rest, $\frac{8\pi}{3} + \sqrt{3}$. The ratio is

$$\frac{\frac{8\pi}{3} + \sqrt{3}}{\frac{4\pi}{3} - \sqrt{3}} = \frac{8\pi + 3\sqrt{3}}{4\pi - 3\sqrt{3}},$$

which has the required form with $(a, b, c, d, e, f) = (8, 3, 3, 4, 3, 3)$.

The product is $8 \cdot 3 \cdot 3 \cdot 4 \cdot 3 \cdot 3 = 2592$, whose remainder upon division by 1000 is 592.

2. A jar has 10 red candies and 10 blue candies. Terry picks two candies at random, then Mary picks two of the remaining candies at random. Given that the probability that they get the same color combination, irrespective of order, is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Solution:

The combinations match exactly when both draw two reds, both draw two blues, or both draw one candy of each color. The probability that Terry draws two reds is $\frac{\binom{10}{2}}{\binom{20}{2}} = \frac{45}{190} = \frac{9}{38}$, after which 8 reds and 10 blues remain, so Mary draws two reds with probability $\frac{\binom{8}{2}}{\binom{18}{2}} = \frac{28}{153}$. That case has probability $\frac{9}{38} \cdot \frac{28}{153} = \frac{14}{323}$, and by symmetry two blues each is also $\frac{14}{323}$.
For mixed draws, Terry succeeds with probability $\frac{10\cdot 10}{\binom{20}{2}} = \frac{10}{19}$, leaving 9 of each color, and Mary with probability $\frac{9\cdot 9}{\binom{18}{2}} = \frac{9}{17}$, for $\frac{10}{19} \cdot \frac{9}{17} = \frac{90}{323}$.
The total is $\frac{14+14+90}{323} = \frac{118}{323}$. Since $118 = 2 \cdot 59$ and $323 = 17 \cdot 19$, the fraction is in lowest terms, and $m+n = 118+323 = 441$.

3. A solid rectangular block is formed by gluing together N congruent 1-cm cubes face to face. When the block is viewed so that three of its faces are visible, exactly 231 of the 1-cm cubes cannot be seen. Find the smallest possible value of N.

Solution:

Let the block measure $p \times q \times r$. A cube is hidden exactly when it touches none of the three visible faces, so the hidden cubes form a $(p-1) \times (q-1) \times (r-1)$ block, giving $(p-1)(q-1)(r-1) = 231 = 3 \cdot 7 \cdot 11$.

The ways to write 231 as a product of three positive integers are $3 \cdot 7 \cdot 11, 1 \cdot 3 \cdot 77, 1 \cdot 7 \cdot 33, 1 \cdot 11 \cdot 21,$ and $1 \cdot 1 \cdot 231,$ giving blocks $4 \times 8 \times 12, 2 \times 4 \times 78, 2 \times 8 \times 34, 2 \times 12 \times 22,$ and $2 \times 2 \times 232,$ with volumes $384, 624, 544, 528,$ and $928.$

The smallest is $N = 384.$

4. How many positive integers less than $10{,}000$ have at most two different digits?

Solution:

All 99 positive integers below 100 qualify. A qualifying 3-digit number is either a repdigit (9 of them) or uses a leading digit $a \geq 1$ together with a second value $b \neq a$ in some of the last two positions: $2^2 - 1 = 3$ patterns, each realized in $9 \cdot 9$ ways (9 choices for a, then 9 for b), for $9 + 3 \cdot 81 = 252$ numbers.

Similarly a qualifying 4-digit number is a repdigit (9) or has $b \neq a$ appearing in a nonempty subset of the last three positions: $2^3 - 1 = 7$ patterns, each in $9 \cdot 9$ ways, for $9 + 7 \cdot 81 = 576$ numbers.

The total is $99 + 252 + 576 = 927.$

5. In order to complete a large job, 1000 workers were hired, just enough to complete the job on schedule. All the workers stayed on the job while the first quarter of the work was done, so the first quarter of the work was completed on schedule. Then 100 workers were laid off, so the second quarter of the work was completed behind schedule. Then an additional 100 workers were laid off, so the third quarter of the work was completed still further behind schedule. Given that all workers work at the same rate, what is the minimum number of additional workers, beyond the 800 workers still on the job at the end of the third quarter, that must be hired after three-quarters of the work has been completed so that the entire project can be completed on schedule or before?

Solution:

Measure time so that 1000 workers complete a quarter of the job in 1 unit; the schedule allows 4 units in all. With 900 workers the second quarter takes $\frac{10}{9}$ units, and with 800 workers the third quarter takes $\frac{10}{8} = \frac{5}{4}$ units. The time used so far is

$$1 + \frac{10}{9} + \frac{5}{4} = \frac{121}{36},$$

leaving $4 - \frac{121}{36} = \frac{23}{36}$ of a unit for the last quarter.

The last quarter requires 1000 worker-units of labor, so the workforce w must satisfy $w \cdot \frac{23}{36} \geq 1000,$ that is $w \geq \frac{36000}{23} \approx 1565.2,$ so at least 1566 workers are needed.

Since 800 remain on the job, at least $1566 - 800 = 766$ additional workers must be hired.

6. Three clever monkeys divide a pile of bananas. The first monkey takes some bananas from the pile, keeps three-fourths of them, and divides the rest equally between the other two. The second monkey takes some bananas from the pile, keeps one-fourth of them, and divides the rest equally between the other two. The third monkey takes the remaining bananas from the pile, keeps one-twelfth of them, and divides the rest equally between the other two. Given that each monkey receives a whole number of bananas whenever the bananas are divided, and the numbers of bananas the first, second, and third monkeys have at the end of the process are in the ratio $3 : 2 : 1$, what is the least possible total for the number of bananas?

Solution:

Say the first monkey takes $8x$ bananas, keeping $6x$ and giving x to each of the others; the second takes $8y$, keeping $2y$ and giving $3y$ to each; the third takes $24z$, keeping $2z$ and giving $11z$ to each. All divisions are whole numbers exactly when x, y, z are positive integers. The final amounts are $6x + 3y + 11z, x + 2y + 11z$, and $x + 3y + 2z$.

The ratio $3 : 2 : 1$ says the first amount is triple the third and the second is double the third:

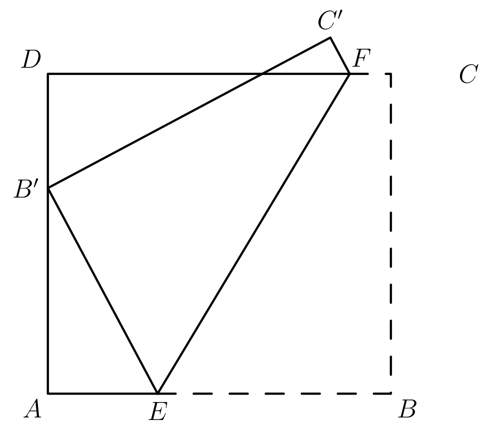
$$6x + 3y + 11z = 3(x + 3y + 2z) \implies 3x + 5z = 6y,$$

$$x + 2y + 11z = 2(x + 3y + 2z) \implies x + 4y = 7z.$$

Substituting $x = 7z - 4y$ into the first equation gives $26z = 18y$, so $9y = 13z$. Thus $y = 13n$ and $z = 9n$ for a positive integer n, and then $x = 63n - 52n = 11n$.

The total is $8x + 8y + 24z = (88 + 104 + 216)n = 408n$, least when $n = 1$: the answer is 408.

7. $ABCD$ is a rectangular sheet of paper that has been folded so that corner B is matched with point B' on edge $\overline{AD}$. The crease is $\overline{EF}$, where E is on $\overline{AB}$ and F is on $\overline{CD}$. The dimensions $AE = 8, BE = 17,$ and $CF = 3$ are given. The perimeter of rectangle $ABCD$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution:

Folding reflects B to B' across the crease, so $B'E = BE = 17$. In right triangle AEB', $AB' = \sqrt{17^2 - 8^2} = 15$, and $AB = AE + EB = 25$. Place $A = (0, 0)$, $B = (25, 0)$, $B' = (0, 15)$.

Points on the crease are equidistant from B and B', so $\overline{EF}$ is perpendicular to $\overline{BB'}$. Since BB' has slope $-\frac{3}{5}$, the crease through $E = (8, 0)$ has slope $\frac{5}{3}$, and it meets the line CD (at height $h = BC$) at $x = 8 + \frac{3h}{5}$. The condition $CF = 3$ gives

$$25 - \left(8 + \frac{3h}{5}\right) = 3, \qquad \text{so} \qquad h = \frac{70}{3}.$$

The perimeter is $2\left(25 + \frac{70}{3}\right) = \frac{290}{3}$, so $m + n = 290 + 3 = 293$.

8. How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

Solution:

Since $2004 = 2^2 \cdot 3 \cdot 167,$ we have $2004^{2004} = 2^{4008} \cdot 3^{2004} \cdot 167^{2004}$, so its divisors are $N = 2^i 3^j 167^k$ with $i \leq 4008$ and $j, k \leq 2004.$ Such an N has $(i+1)(j+1)(k+1)$ divisors, so we need $(i+1)(j+1)(k+1) = 2004.$

Every ordered triple of positive integers with product 2004 yields admissible exponents, since each factor is at most $2004.$ Counting prime by prime: the exponent 2 of the prime 2 is split among the three factors in $\binom{2+2}{2} = 6$ ways by stars and bars, and each of the primes 3 and 167 goes to one of the 3 factors.

The count is $6 \cdot 3 \cdot 3 = 54.$

9. A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms a_{2n-1}, a_{2n}, and a_{2n+1} are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

Solution:

Let $a_2 = r$. The geometric condition gives $a_3 = r^2$, the arithmetic condition gives $a_4 = 2r^2 - r = r(2r-1)$, then $a_5 = (2r-1)^2$, and so on: inductively

$$a_{2k+1} = \big(kr - (k-1)\big)^2, \qquad a_{2k+2} = \big(kr - (k-1)\big)\big((k+1)r - k\big).$$

In particular $a_9 = (4r-3)^2$ and $a_{10} = (4r-3)(5r-4)$, so $a_9 + a_{10} = (4r-3)(9r-7) = 646$. Expanding gives $36r^2 - 55r - 625 = 0$, which factors as $(r-5)(36r+125) = 0$, so $r = 5$.

With $r = 5$ we get $kr - (k-1) = 4k+1$, so $a_{2k+1} = (4k+1)^2$ and $a_{2k+2} = (4k+1)(4k+5)$; the sequence is increasing. Since $a_{17} = 33^2 = 1089 > 1000$ while $a_{16} = 29 \cdot 33 = 957$, the greatest term below 1000 is $a_{16} = 957$.

Therefore $n + a_n = 16 + 957 = 973$.

10. Let $\mathcal{S}$ be the set of integers between 1 and 2^{40} whose binary expansions have exactly two 1's. If a number is chosen at random from $\mathcal{S}$, the probability that it is divisible by 9 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

The set $\mathcal{S}$ consists of the $\binom{40}{2} = 780$ numbers $2^a + 2^b$ with $0 \leq a < b \leq 39$. Since 2^a is coprime to 9, we have $9 \mid 2^a(2^{b-a}+1)$ exactly when $2^{b-a} \equiv -1 \pmod 9$. The powers of 2 modulo 9 cycle through $2, 4, 8, 7, 5, 1$ with period 6, so $2^d \equiv 8 \equiv -1$ exactly when $d \equiv 3 \pmod 6$.

For each difference $d = b - a$ there are $40 - d$ pairs, so the number of multiples of 9 in $\mathcal{S}$ is

$$\sum_{d=3,9,...,39} (40-d) = 37 + 31 + 25 + 19 + 13 + 7 + 1 = 133.$$

The probability is $\frac{133}{780}$, and since $133 = 7 \cdot 19$ while $780 = 2^2 \cdot 3 \cdot 5 \cdot 13$, it is in lowest terms. Thus $p + q = 133 + 780 = 913$.

11. A right circular cone has a base with radius 600 and height $200\sqrt{7}$. A fly starts at a point on the surface of the cone whose distance from the vertex of the cone is 125, and crawls along the surface of the cone to a point on the exact opposite side of the cone whose distance from the vertex is $375\sqrt{2}$. Find the least distance that the fly could have crawled.

Solution:

The slant height is $\sqrt{600^2 + (200\sqrt{7})^2} = \sqrt{360000 + 280000} = 800$. Cutting the cone along the ruling through the starting point and unrolling gives a sector of radius 800 whose arc has the base circumference $2\pi \cdot 600 = 1200\pi$; since a full circle of radius 800 has circumference 1600π, the central angle is $\frac{3}{4} \cdot 360^\circ = 270^\circ$. A point on the exact opposite side of the cone is halfway around, which in the unrolled sector is 135° away.

The shortest crawl is the straight segment between the two points, at radii 125 and $375\sqrt{2}$ with a 135° angle between them. By the law of cosines,

$$d^2 = 125^2 + (375\sqrt{2})^2 - 2 \cdot 125 \cdot 375\sqrt{2}\cos 135^\circ = 15625 + 281250 + 93750 = 390625.$$

Thus $d = 625$.

12. Let $ABCD$ be an isosceles trapezoid, whose dimensions are $AB = 6, BC = 5 = DA,$ and $CD = 4.$ Draw circles of radius 3 centered at A and $B,$ and circles of radius 2 centered at C and $D.$ A circle contained within the trapezoid is tangent to all four of these circles. Its radius is $\frac{-k+m\sqrt{n}}{p},$ where $k, m, n,$ and p are positive integers, n is not divisible by the square of any prime, and k and p are relatively prime. Find $k+m+n+p.$

Solution:

Dropping perpendiculars from C and D shows each leg of length 5 spans a horizontal offset of $\frac{6-4}{2} = 1,$ so the height of the trapezoid is $\sqrt{25-1} = \sqrt{24}.$ By symmetry the inner circle's center O lies on the vertical axis through the midpoints E of $\overline{AB}$ and F of $\overline{CD}.$ If its radius is $x,$ external tangency gives $OA = x+3$ and $OC = x+2,$ so with $AE = 3$ and $CF = 2,$

$$OE = \sqrt{(x+3)^2 - 9} = \sqrt{x^2+6x}, \qquad OF = \sqrt{(x+2)^2 - 4} = \sqrt{x^2+4x}.$$

Since $OE + OF = \sqrt{24},$ moving one radical across and squaring gives $\sqrt{24(x^2+4x)} = 12 - x,$ and squaring again yields $24x^2 + 96x = 144 - 24x + x^2,$ that is $23x^2 + 120x - 144 = 0.$

The positive root is

$$x = \frac{-120 + \sqrt{14400 + 13248}}{46} = \frac{-120 + 96\sqrt{3}}{46} = \frac{-60 + 48\sqrt{3}}{23},$$

so $k+m+n+p = 60 + 48 + 3 + 23 = 134.$

13. Let $ABCDE$ be a convex pentagon with $\overline{AB} \parallel \overline{CE}, \overline{BC} \parallel \overline{AD}, \overline{AC} \parallel \overline{DE}, \angle ABC = 120^\circ, AB = 3, BC = 5$, and $DE = 15$. Given that the ratio between the area of triangle ABC and the area of triangle EBD is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Solution:

By the law of cosines, $AC^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cos 120^\circ = 49$, so $AC = 7$. Let F be the intersection of $\overline{AD}$ and $\overline{CE}$. Since $AF \parallel BC$ and $CF \parallel AB$, quadrilateral $ABCF$ is a parallelogram, so F lies at the same distance h from line AC as B, on the opposite side, where $[ABC] = \frac{1}{2} \cdot 7h$.

Since $\overline{AC} \parallel \overline{DE}$, triangles FAC and FDE are similar with ratio $AC : DE = 7 : 15$, so the distance from F to line DE is $\frac{15h}{7}$, with DE on the far side of F from AC. The distance from B to line DE is therefore $h + h + \frac{15h}{7} = \frac{29h}{7}$, giving

$$[EBD] = \frac{1}{2} \cdot 15 \cdot \frac{29h}{7} = \frac{435h}{14}.$$

Thus $\frac{[ABC]}{[EBD]} = \frac{7h/2}{435h/14} = \frac{49}{435}$, which is in lowest terms since $435 = 3 \cdot 5 \cdot 29$. The answer is $m + n = 49 + 435 = 484$.

14. Consider a string of n 7's, $7777\ldots77$, into which $+$ signs are inserted to produce an arithmetic expression. For example, $7+77+777+7+7=875$ could be obtained from eight 7's in this way. For how many values of n is it possible to insert $+$ signs so that the resulting expression has value 7000?

Solution:

Dividing by 7 turns the summands into $1, 11,$ or 111 (no longer summand fits, since $1111 > 1000$) and the target into 1000. If x, y, z count the summands of each size, then $x + 11y + 111z = 1000$ and $n = x + 2y + 3z$. Subtracting gives

$$n = 1000 - 9(y + 12z),$$

so the possible n correspond exactly to the attainable values of $v = y + 12z$.

The constraints are $z \leq 9$ and $0 \leq y \leq \frac{1000-111z}{11}$ (then $x \geq 0$ follows). For $z = 0, 1, \ldots, 9$ the value $v = y + 12z$ ranges over the intervals $[0, 90]$, $[12, 92]$, $[24, 94]$, $[36, 96]$, $[48, 98]$, $[60, 100]$, $[72, 102]$, $[84, 104]$, $[96, 106]$, and $\{108\}$ (when $z = 9$, only $y = 0$ fits). Their union is every integer from 0 to 106 together with 108; only 107 is unattainable.

So v takes $107 + 1 = 108$ values, and $n = 1000 - 9v$ takes 108 values.

15. A long thin strip of paper is 1024 units in length, 1 unit in width, and is divided into 1024 unit squares. The paper is folded in half repeatedly. For the first fold, the right end of the paper is folded over to coincide with and lie on top of the left end. The result is a 512 by 1 strip of double thickness. Next, the right end of this strip is folded over to coincide with and lie on top of the left end, resulting in a 256 by 1 strip of quadruple thickness. This process is repeated 8 more times. After the last fold, the strip has become a stack of 1024 unit squares. How many of these squares lie below the square that was originally the 942nd square counting from the left?

Solution:

After f folds the strip is 2^{10-f} squares long and 2^f layers thick, so the positions L from the left and R from the right satisfy $L+R=2^{10-f}+1$, and the positions B from the bottom and T from the top satisfy $B+T=2^f+1$. When the right half is folded over onto the left, a square in the left half keeps its L and B, while a square in the right half is flipped: its new L is its old R, and its new T is its old B.

The 942nd square starts at $(L,B)=(942,1)$. Applying the rule through the ten folds gives

$$(83,2),\ (83,2),\ (83,2),\ (46,15),\ (19,18),\ (14,47),\ (3,82),\ (3,82),\ (2,431),\ (1,594).$$

For example, at the fourth fold the strip has length 128 and $L=83>64$, so the new L is $128+1-83=46$ and the new T is the old $B=2$, making $B=16+1-2=15$.

In the final stack of 1024 squares, this square sits at height 594 from the bottom, so $594-1=593$ squares lie below it.

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