2003 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Given that $\frac{((3!)!)!}{3!} = k \cdot n!$, where k and n are positive integers and n is as large as possible, find $k + n$.

Solution:

Since $3! = 6$ and $6! = 720$, the expression is

$$\frac{((3!)!)!}{3!} = \frac{720!}{6} = \frac{720 \cdot 719!}{6} = 120 \cdot 719!.$$

If n were 720 or more, then $k \cdot n! \geq 720!$, which exceeds $\frac{720!}{6}$. So the largest possible value of n is 719, achieved with $k = 120$, and $k + n = 120 + 719 = 839$.

2. One hundred concentric circles with radii $1, 2, 3, \ldots, 100$ are drawn in a plane. The interior of the circle of radius 1 is colored red, and each region bounded by consecutive circles is colored either red or green, with no two adjacent regions the same color. The ratio of the total area of the green regions to the area of the circle of radius 100 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

The regions alternate red, green, red, green, . . . from the center outward, so the green regions are the annuli between radii 1 and 2, between 3 and 4, and so on up to the annulus between 99 and 100. Their total area is

$$\pi \left[(2^2 - 1^2) + (4^2 - 3^2) + \cdots + (100^2 - 99^2)\right] = \pi \left[(2+1) + (4+3) + \cdots + (100+99)\right],$$

which is $\pi\,(1 + 2 + \cdots + 100) = 5050\pi$.

The desired ratio is $\frac{5050\pi}{100^2\pi} = \frac{101}{200}$, so $m + n = 101 + 200 = 301$.

3. Let the set $\mathcal{S} = \{8, 5, 1, 13, 34, 3, 21, 2\}$. Susan makes a list as follows: for each two-element subset of $\mathcal{S}$, she writes on her list the greater of the set's two elements. Find the sum of the numbers on the list.

Solution:

An element x is the greater element of a two-element subset exactly once for each smaller element of the set, so x contributes to the sum once per element below it. Sorting the set as $1, 2, 3, 5, 8, 13, 21, 34$, the sum of the list is

$$0(1) + 1(2) + 2(3) + 3(5) + 4(8) + 5(13) + 6(21) + 7(34) = 2 + 6 + 15 + 32 + 65 + 126 + 238 = 484.$$

4. Given that $\log_{10} \sin x + \log_{10} \cos x = -1$ and that $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, find n.

Solution:

The first equation says $\log_{10}(\sin x \cos x) = -1$, so $\sin x \cos x = \frac{1}{10}$. Then

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \frac{2}{10} = \frac{12}{10}.$$

Taking logarithms, $2 \log_{10}(\sin x + \cos x) = \log_{10} \frac{12}{10} = \log_{10} 12 - 1$, so $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} 12 - 1)$ and $n = 12$.

5. Consider the set of points that are inside or within one unit of a rectangular parallelepiped (box) that measures 3 by 4 by 5 units. Given that the volume of this set is $\frac{m+n\pi}{p}$, where $m, n,$ and p are positive integers, and n and p are relatively prime, find $m+n+p$.

Solution:

The region consists of the box itself, six slabs of thickness 1 projecting outward from the faces, quarter-cylinders of radius 1 along the twelve edges, and eighth-spheres of radius 1 at the eight corners. The box has volume $3 \cdot 4 \cdot 5 = 60$, and the slabs total $2(3 \cdot 4 + 3 \cdot 5 + 4 \cdot 5) = 94$.

The four quarter-cylinders along edges parallel to each dimension combine into a full cylinder, so the cylinders total $\pi \cdot 1^2(3+4+5) = 12\pi$. The eight octants combine into one unit sphere of volume $\frac{4\pi}{3}$.

The total volume is

$$60 + 94 + 12\pi + \frac{4\pi}{3} = 154 + \frac{40\pi}{3} = \frac{462 + 40\pi}{3},$$

so $m+n+p = 462 + 40 + 3 = 505.$

6. The sum of the areas of all triangles whose vertices are also vertices of a 1 by 1 by 1 cube is $m + \sqrt{n} + \sqrt{p}$, where $m, n,$ and p are integers. Find $m+n+p$.

Solution:

Every side of such a triangle is a cube edge, a face diagonal of length $\sqrt{2}$, or a space diagonal of length $\sqrt{3}$. Only three shapes occur. A triangle of two adjacent edges and a face diagonal is right with area $\frac{1}{2}$; there are 4 per face, or 24. A triangle of three face diagonals is equilateral with area $\frac{\sqrt{3}}{2}$; each is determined by the three vertices adjacent to one of the 8 cube vertices, so there are 8. A triangle of an edge, a face diagonal, and a space diagonal is right with legs 1 and $\sqrt{2}$, so its area is $\frac{\sqrt{2}}{2}$; each of the 4 space diagonals forms one with each of the 6 vertices off that diagonal, so there are 24. (Indeed $24 + 8 + 24 = \binom{8}{3} = 56$.)

The total area is

$$24 \cdot \frac{1}{2} + 8 \cdot \frac{\sqrt{3}}{2} + 24 \cdot \frac{\sqrt{2}}{2} = 12 + 4\sqrt{3} + 12\sqrt{2} = 12 + \sqrt{48} + \sqrt{288},$$

so $m+n+p = 12 + 48 + 288 = 348.$

7. Point B is on $\overline{AC}$ with $AB = 9$ and $BC = 21$. Point D is not on $\overline{AC}$ so that $AD = CD$, and AD and BD are integers. Let s be the sum of all possible perimeters of $\triangle ACD$. Find s.

Solution:

Let $AD = CD = a$ and $BD = b$, and let E be the foot of the perpendicular from D to $\overline{AC}$. Since $AD = CD$, point E is the midpoint of $\overline{AC}$, so $AE = 15$ and $BE = 15 - 9 = 6$. The right triangles DEA and DEB share leg DE, so

$$a^2 - 15^2 = DE^2 = b^2 - 6^2, \qquad \text{that is} \qquad (a+b)(a-b) = 189.$$

The factorizations $189 = 189 \cdot 1 = 63 \cdot 3 = 27 \cdot 7 = 21 \cdot 9$ give $(a, b) = (95, 94), (33, 30), (17, 10)$, and $(15, 6)$. The last is rejected: $b = 6$ would put D on $\overline{AC}$. Each valid pair gives a triangle with perimeter $2a + 30$.

Therefore $s = (190 + 30) + (66 + 30) + (34 + 30) = 220 + 96 + 64 = 380.$

8. In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.

Solution:

Write the terms as $a, a+d, a+2d$, and $a+30$, where a and d are positive integers. The geometric condition on the last three terms says $(a+30)(a+d) = (a+2d)^2$. Expanding both sides and simplifying,

$$30a + 30d = 3ad + 4d^2, \qquad \text{that is} \qquad 3a(10-d) = 2d(2d-15).$$

Since $a, d > 0$, the factors $10 - d$ and $2d - 15$ must have the same sign, forcing $7.5 < d < 10$, so $d = 8$ or $d = 9$. For $d = 8$, we get $6a = 16$, which has no integer solution. For $d = 9$, we get $3a = 54$, so $a = 18$.

The sequence is $18, 27, 36, 48$ (indeed $27, 36, 48$ has ratio $\frac{4}{3}$), and the sum is $18 + 27 + 36 + 48 = 129$.

9. An integer between 1000 and 9999, inclusive, is called *balanced* if the sum of its two leftmost digits equals the sum of its two rightmost digits. How many balanced integers are there?

Solution:

Group the balanced integers by the common sum s of each digit pair, where $1 \le s \le 18$. For $s \le 9$, the leftmost pair (first digit at least 1) can be formed in s ways and the rightmost pair in $s+1$ ways. For $s \ge 10$, both digits of each pair must be at least $s-9$, giving $19-s$ ways for each pair.

The total count is

$$\sum_{s=1}^{9} s(s+1) + \sum_{s=10}^{18} (19-s)^2 = \sum_{s=1}^{9} (s^2+s) + \sum_{k=1}^{9} k^2 = 2 \cdot 285 + 45 = 615.$$

10. Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106^\circ$. Point M is in the interior of the triangle so that $\angle MAC = 7^\circ$ and $\angle MCA = 23^\circ$. Find the number of degrees in $\angle CMB$.

Solution:

Assume $AC = BC = 1$. In triangle AMC, the angles at A and C are 7° and 23°, so $\angle AMC = 150^\circ$, and the Law of Sines gives

$$CM = \frac{\sin 7^\circ}{\sin 150^\circ} = 2\sin 7^\circ.$$

Also $\angle MCB = 106^\circ - 23^\circ = 83^\circ$, whose cosine is $\sin 7^\circ$. The Law of Cosines in triangle BMC then gives

$$MB^2 = CM^2 + CB^2 - 2 \cdot CM \cdot CB \cos 83^\circ = 4\sin^2 7^\circ + 1 - 4\sin^2 7^\circ = 1.$$

So $MB = 1 = CB$, making triangle BMC isosceles with $\angle CMB = \angle MCB = 83^\circ$. The answer is 83.

11. An angle x is chosen at random from the interval $0^\circ < x < 90^\circ$. Let p be the probability that the numbers $\sin^2 x, \cos^2 x$, and $\sin x \cos x$ are *not* the lengths of the sides of a triangle. Given that $p = \frac{d}{n}$, where d is the number of degrees in $\arctan m$ and m and n are positive integers with $m + n < 1000$, find $m + n$.

Solution:

Replacing x by $90^\circ - x$ swaps $\sin x$ and $\cos x$, so the failure probability on $(45^\circ, 90^\circ)$ matches that on $(0^\circ, 45^\circ)$, and it suffices to consider $0^\circ < x \le 45^\circ$. There $\cos^2 x \ge \sin x \cos x \ge \sin^2 x$, so the three numbers fail to form a triangle exactly when

$$\cos^2 x \ge \sin^2 x + \sin x \cos x.$$

Since $\cos^2 x - \sin^2 x = \cos 2x$ and $\sin x \cos x = \frac{1}{2} \sin 2x$, this says $\cos 2x \ge \frac{1}{2} \sin 2x$, i.e. $\tan 2x \le 2$. Because tangent increases on this range, that happens exactly for $x \le \frac{1}{2} \arctan 2$.

Therefore

$$p = \frac{\frac{1}{2} \arctan 2}{45^\circ} = \frac{\arctan 2}{90^\circ},$$

so $m = 2$ and $n = 90$, with $m + n = 92 < 1000$, and the answer is 92.

12. In convex quadrilateral $ABCD, \angle A \cong \angle C, AB = CD = 180$, and $AD \ne BC$. The perimeter of $ABCD$ is 640. Find $\lfloor 1000 \cos A \rfloor$. (The notation $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.)

Solution:

Let $\angle A = \angle C = \alpha, AD = x$, and $BC = y$. Applying the Law of Cosines to diagonal BD in triangles ABD and CDB,

$$BD^2 = x^2 + 180^2 - 2 \cdot 180x \cos \alpha = y^2 + 180^2 - 2 \cdot 180y \cos \alpha.$$

Rearranging gives $x^2 - y^2 = 2 \cdot 180(x - y) \cos \alpha$, and since $x \ne y$ we may divide by $x - y$:

$$\cos \alpha = \frac{x + y}{360} = \frac{640 - 2 \cdot 180}{360} = \frac{280}{360} = \frac{7}{9}.$$

Then $1000 \cos A = \frac{7000}{9} = 777.7\ldots$, so $\lfloor 1000 \cos A \rfloor = 777$.

13. Let N be the number of positive integers that are less than or equal to 2003 and whose base-2 representation has more 1's than 0's. Find the remainder when N is divided by 1000.

Solution:

Since $2003 < 2^{11} = 2048$, every integer in question has at most 11 binary digits. A $(d+1)$-digit binary number starts with 1, and choosing k more 1's among the remaining d digits gives $\binom{d}{k}$ numbers with $k+1$ ones; the 1's outnumber the 0's exactly when $k \geq \frac{d}{2}$. So the count over all numbers up to 2047 is the total of the entries on or to the right of the center of rows 0 through 10 of Pascal's triangle.

Those rows sum to $1 + 2 + \cdots + 2^{10} = 2047$, and the central entries sum to $\sum_{i=0}^{5} \binom{2i}{i} = 1 + 2 + 6 + 20 + 70 + 252 = 351$, so by symmetry the count is $\frac{2047+351}{2} = 1199$.

The 44 integers from 2004 to 2047 all exceed $1984 = 11111000000_2$, so each has the prefix 11111 plus at least one more 1, hence at least six 1's among eleven digits — all 44 were counted. Therefore $N = 1199 - 44 = 1155$, whose remainder upon division by 1000 is 155.

14. The decimal representation of $\frac{m}{n}$, where m and n are relatively prime positive integers and $m < n$, contains the digits $2, 5,$ and 1 consecutively, and in that order. Find the smallest value of n for which this is possible.

Solution:

It suffices to make 251 appear immediately after the decimal point: if $\frac{m}{n} = .A251\ldots$ with A a block of $k \geq 1$ digits, then $10^k \frac{m}{n} - A = .251\ldots$ is a fraction between 0 and 1 whose reduced denominator is at most n. So we need the smallest n admitting an m with

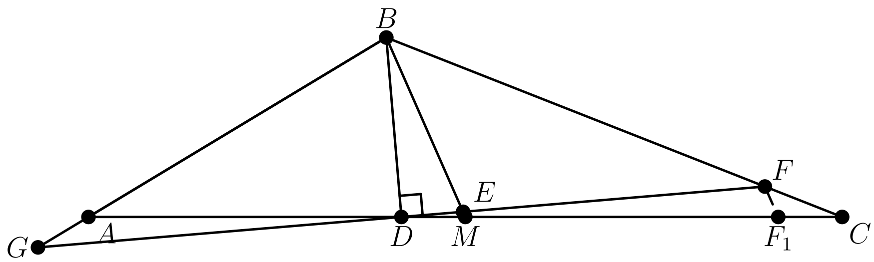
$$\frac{251}{1000} \leq \frac{m}{n} < \frac{252}{1000}, \qquad \text{that is} \qquad 0 \leq 1000m - 251n < n.$$

Thus $251n$ must land within n below a multiple of 1000. Try $n = 4m - 1$: then $251n = 251(4m-1) = 1000m + (4m - 251)$, so for $m \leq 62$ this lies below $1000m$ by $251 - 4m$. The requirement $251 - 4m < n = 4m - 1$ gives $m > 31.5$, so $m = 32$ and $n = 127$ work: indeed $\frac{32}{127} = .2519\ldots$ A short check of the same inequality shows no smaller n puts $1000m$ within n above a multiple of 251, since the deficit $251 - 4m$ (or its analogues for other residues) stays too large.

The smallest possible value of n is 127.

15. In $\triangle ABC$, $AB = 360$, $BC = 507$, and $CA = 780$. Let M be the midpoint of $\overline{CA}$, and let D be the point on $\overline{CA}$ such that $\overline{BD}$ bisects angle ABC. Let F be the point on $\overline{BC}$ such that $\overline{DF} \perp \overline{BD}$. Suppose that $\overline{DF}$ meets $\overline{BM}$ at E. The ratio $DE : EF$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:



Write $c = AB = 360$, $a = BC = 507$, $b = CA = 780$. Extend $\overline{FD}$ beyond D to meet ray BA beyond A at G. In triangle BGF, segment $\overline{BD}$ is both an angle bisector and an altitude, so $BG = BF = t$. The bisector also gives $\frac{CD}{DA} = \frac{a}{c}$, so Menelaus' theorem for line GDF crossing triangle ABC says

$$\frac{AG}{GB} \cdot \frac{BF}{FC} \cdot \frac{CD}{DA} = \frac{t-c}{t} \cdot \frac{t}{a-t} \cdot \frac{a}{c} = 1, \qquad \text{so} \qquad t = \frac{2ac}{a+c}.$$

Now let F_1 be the point on $\overline{AC}$ with $\overline{FF_1} \parallel \overline{BM}$. Since E lies on $\overline{BM}$, we have $\overline{EM} \parallel \overline{FF_1}$, so triangles DEM and DFF_1 are similar and $\frac{DE}{EF} = \frac{DM}{MF_1}$. The bisector ratio gives $AD = \frac{bc}{a+c}$, so $DM = \frac{b}{2} - \frac{bc}{a+c} = \frac{b(a-c)}{2(a+c)}$. Also $CF = a - t = \frac{a(a-c)}{a+c}$, so $CF_1 = CM \cdot \frac{CF}{CB} = \frac{b}{2} \cdot \frac{a-c}{a+c}$ and $MF_1 = \frac{b}{2}\left(1 - \frac{a-c}{a+c}\right) = \frac{bc}{a+c}$.

Therefore

$$\frac{DE}{EF} = \frac{DM}{MF_1} = \frac{a-c}{2c} = \frac{147}{720} = \frac{49}{240},$$

and $m + n = 49 + 240 = 289$.

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