2002 AIME II Solutions

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1. Given that

(1) x and y are both integers between 100 and 999, inclusive;

(2) y is the number formed by reversing the digits of x; and

(3) $z = |x - y|$.

How many distinct values of z are possible?

Solution:

Write $x = 100h + 10t + u$ with digits h, t, u. Then $y = 100u + 10t + h$, so

$$z = |x - y| = 99\,|h - u|.$$

Since both x and y are three-digit numbers, both h and u run from 1 to 9, so $|h - u|$ can be any of $0, 1, \ldots, 8$. Each choice gives a different multiple of 99, so there are 9 distinct values of z.

2. Three of the vertices of a cube are $P = (7, 12, 10)$, $Q = (8, 8, 1)$, and $R = (11, 3, 9)$. What is the surface area of the cube?

Solution:

Compute the squared distances: $PQ^2 = 1^2 + 4^2 + 9^2 = 98$, $QR^2 = 3^2 + 5^2 + 8^2 = 98$, and $RP^2 = 4^2 + 9^2 + 1^2 = 98$. So P, Q, and R form an equilateral triangle with side $\sqrt{98} = 7\sqrt{2}$.

Three mutually equidistant vertices of a cube must be joined by face diagonals, and a face diagonal of a cube with edge s has length $s\sqrt{2}$. Thus $s = 7$, and the surface area is $6 \cdot 7^2 = 294$.

3. It is given that $\log_6 a + \log_6 b + \log_6 c = 6$, where a, b, and c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

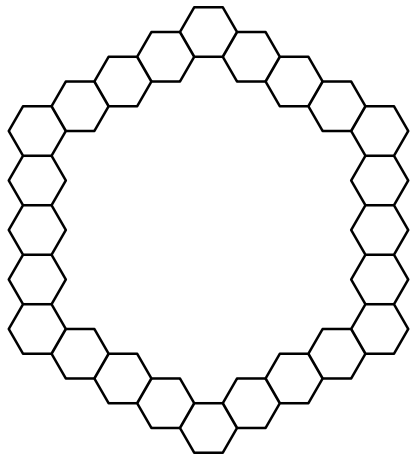
Solution:

Adding the logs gives $\log_6(abc) = 6$, so $abc = 6^6$. In a geometric sequence $ac = b^2$, hence $b^3 = 6^6$, so $b = 36$ and $ac = 36^2 = 1296$.

Since the sequence is increasing, $b - a$ is a positive perfect square, so $a = 36 - k^2$ for some $k = 1, \ldots, 5$, giving candidates $35, 32, 27, 20, 11$. Also a must divide $1296 = 2^4 \cdot 3^4$, and of the candidates only 27 does, with $c = 1296/27 = 48$.

Indeed $27, 36, 48$ is geometric with ratio $\frac{4}{3}$, and $a + b + c = 27 + 36 + 48 = 111$.

4. Patio blocks that are regular hexagons 1 unit on a side are used to outline a garden by placing the blocks edge to edge with n on each side. The diagram indicates the path of blocks around the garden when $n = 5$.



If $n = 202$, then the area of the garden enclosed by the path, not including the path itself, is $m\left(\sqrt{3}/2\right)$ square units, where m is a positive integer. Find the remainder when m is divided by 1000.

Solution:

The garden enclosed by the path is itself a hexagonal arrangement of unit hexagons with $n-1$ on each side. Counting from the center outward in rings of $6, 12, \ldots$ hexagons, it contains

$$1 + 6 + 12 + \cdots + 6(n-2) = 1 + 3(n-2)(n-1)$$

blocks, which for $n = 202$ is $1 + 3 \cdot 200 \cdot 201 = 120601$.

Each unit hexagon consists of 6 equilateral triangles of side 1, so its area is $6 \cdot \frac{\sqrt{3}}{4} = 3 \cdot \frac{\sqrt{3}}{2}$. The garden's area is therefore $3 \cdot 120601 = 361803$ times $\frac{\sqrt{3}}{2}$, so $m = 361803$, and the remainder upon division by 1000 is 803.

5. Find the sum of all positive integers $a = 2^n 3^m$, where n and m are non-negative integers, for which a^6 is not a divisor of 6^a.

Solution:

With $a = 2^n 3^m$,

$$\frac{6^a}{a^6} = \frac{2^a 3^a}{2^{6n} 3^{6m}},$$

which fails to be an integer exactly when $6n > a$ or $6m > a$.

If $m, n \geq 1$, then $a \geq 3 \cdot 2^n \geq 6n$ (since $2^n \geq 2n$) and similarly $a \geq 2 \cdot 3^m \geq 6m$, so no such a works. If $m = 0$, the condition is $2^n < 6n$, which holds for $n = 1, 2, 3, 4$, giving $a = 2, 4, 8, 16$. If $n = 0$, the condition is $3^m < 6m$, which holds for $m = 1, 2$, giving $a = 3, 9$. (For $a = 1$ the condition fails.)

The sum is $2 + 4 + 8 + 16 + 3 + 9 = 42$.

6. Find the integer that is closest to

$$1000 \sum_{n=3}^{10000} \frac{1}{n^2 - 4}.$$

Solution:

Since $\frac{1}{n^2-4} = \frac{1}{4}\left(\frac{1}{n-2} - \frac{1}{n+2}\right)$, the sum telescopes:

$$1000 \sum_{n=3}^{10000} \frac{1}{n^2 - 4} = 250\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{9999} - \frac{1}{10000} - \frac{1}{10001} - \frac{1}{10002}\right).$$

The front part is $250 \cdot \frac{25}{12} = 520.8\overline{3}$, and the four tail terms subtract only about $250 \cdot \frac{4}{10000} = 0.1$. The value is therefore about 520.73, so the closest integer is 521.

7. It is known that, for all positive integers k,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Find the smallest positive integer k such that $1^2 + 2^2 + 3^2 + \cdots + k^2$ is a multiple of 200.

Solution:

The sum is a multiple of 200 exactly when $k(k+1)(2k+1)$ is a multiple of $1200 = 2^4 \cdot 3 \cdot 5^2$. The factor 3 always divides $k(k+1)(2k+1)$ (if $k \equiv 1 \pmod 3$, then $3 \mid 2k+1$), so only 2^4 and 5^2 matter.

Since $2k+1$ is odd and $k, k+1$ cannot both be even, 16 must divide k or $k+1$, so $k \equiv 0$ or $15 \pmod{16}$. Similarly 25 must divide one of $k, k+1, 2k+1$, giving $k \equiv 0$, 24, or $12 \pmod{25}$. Combining each pair of congruences modulo 400, the smallest positive solutions are $112, 175, 224, 287, 399$, and 400.

The least is $k = 112$: indeed $112 \cdot 113 \cdot 225 = (16 \cdot 7) \cdot 113 \cdot (9 \cdot 25)$ is a multiple of 1200.

8. Find the least positive integer k for which the equation $\left\lfloor \frac{2002}{n} \right\rfloor = k$ has no integer solutions for n. (The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x.)

Solution:

The value k is attained exactly when some integer n satisfies $k \le \frac{2002}{n} < k+1$, that is, when the interval $\left(\frac{2002}{k+1}, \frac{2002}{k}\right]$ contains an integer. Its length is $\frac{2002}{k(k+1)}$, which is at least 1 whenever $k(k+1) \le 2002$ — so every $k \le 44$ is attained.

For larger k, check directly: $n = 44, 43, 42, 41, 40$ give $\left\lfloor \frac{2002}{n} \right\rfloor = 45, 46, 47, 48, 50.$ Since $\frac{2002}{41} \approx 48.8$ and $\frac{2002}{40} > 50$, the value 49 is never attained, so the least such k is $49.$

9. Let $\mathcal{S}$ be the set $\{1, 2, 3, \ldots, 10\}$. Let n be the number of sets of two non-empty disjoint subsets of $\mathcal{S}$. (*Disjoint sets* are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000.

Solution:

Count ordered pairs (A, B) of disjoint subsets first: each of the 10 elements goes in A, in B, or in neither, for 3^{10} pairs. Among these, 2^{10} have A empty and 2^{10} have B empty, with the pair $(\varnothing, \varnothing)$ counted in both, so

$$3^{10} - 2 \cdot 2^{10} + 1 = 57002$$

ordered pairs have both subsets non-empty.

Disjoint non-empty subsets are never equal, so each set $\{A, B\}$ is counted twice, giving $n = \frac{57002}{2} = 28501$. The remainder mod 1000 is 501.

10. While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of x for which the sine of x degrees is the same as the sine of x radians are $\frac{m\pi}{n-\pi}$ and $\frac{p\pi}{q+\pi}$, where m, n, p, and q are positive integers. Find $m+n+p+q$.

Solution:

An angle of x degrees is $\frac{\pi x}{180}$ radians, so we need $\sin \frac{\pi x}{180} = \sin x$. Two angles have equal sines exactly when they differ by a multiple of 2π or sum to π plus a multiple of 2π.

The first case gives $x - \frac{\pi x}{180} = 2\pi j$, so $x = \frac{360 j\pi}{180-\pi}$, with least positive value $\frac{360\pi}{180-\pi} \approx 6.4$. The second gives $x + \frac{\pi x}{180} = (2k+1)\pi$, so $x = \frac{180(2k+1)\pi}{180+\pi}$, with least positive value $\frac{180\pi}{180+\pi} \approx 3.1$. These are the two smallest solutions.

Matching $\frac{m\pi}{n-\pi}$ and $\frac{p\pi}{q+\pi}$ gives $m = 360, n = 180, p = 180, q = 180$, so $m+n+p+q = 900$.

11. Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is $\frac{1}{8}$, and the second term of both series can be written in the form $\frac{\sqrt{m}-n}{p}$, where m, n, and p are positive integers and m is not divisible by the square of any prime. Find $100m + 10n + p$.

Solution:

A geometric series with ratio r and sum 1 has first term $1 - r$, so its second term is $r(1-r)$. If the two ratios are r and s, then $r(1-r) = s(1-s)$ gives $r - s = r^2 - s^2$, and since the series are distinct, $r \neq s$, forcing $s = 1 - r$.

Say the series with ratio r has third term $r^2(1-r) = \frac{1}{8}$, i.e. $8r^3 - 8r^2 + 1 = 0$. Substituting $t = 2r$ gives $t^3 - 2t^2 + 1 = (t-1)(t^2 - t - 1) = 0$. The root $t = 1$ makes $r = s = \frac{1}{2}$ (the series would coincide), and $r = \frac{1-\sqrt{5}}{4}$ forces $s = 1 - r > 1$, which diverges. So $r = \frac{1+\sqrt{5}}{4}$.

The common second term is

$$r(1-r) = \frac{1+\sqrt{5}}{4} \cdot \frac{3-\sqrt{5}}{4} = \frac{2\sqrt{5}-2}{16} = \frac{\sqrt{5}-1}{8},$$

so $m = 5, n = 1, p = 8$, and $100m + 10n + p = 518$.

12. A basketball player has a constant probability of $.4$ of making any given shot, independent of previous shots. Let a_n be the ratio of shots made to shots attempted after n shots. The probability that $a_{10} = .4$ and $a_n \leq .4$ for all n such that $1 \leq n \leq 9$ is given to be $p^a q^b r / \left(s^c\right)$, where $p, q, r,$ and s are primes, and $a, b,$ and c are positive integers. Find $(p+q+r+s)(a+b+c)$.

Solution:

Record the player's progress as a path through points (n, y), where y is the number of shots made after n attempts. The condition $a_n \leq .4$ caps y at $\lfloor 0.4n \rfloor$, which for $n = 1, \ldots, 9$ is $0, 0, 1, 1, 2, 2, 2, 3, 3,$ and $a_{10} = .4$ means the path ends at $(10, 4)$.

Count the allowed paths by adding, at each point, the counts of its two predecessors (a miss keeps y, a make raises it by 1). The counts at the maximum allowed heights for $n = 3, \ldots, 9$ come out to $1, 2, 2, 5, 9, 9, 23,$ and the tenth shot must be a make, so 23 shot sequences qualify. Each consists of 4 makes and 6 misses, so the probability is

$$23\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^6 = \frac{2^4\, 3^6 \cdot 23}{5^{10}}.$$

Thus $\{p, q, r, s\} = \{2, 3, 23, 5\}$ and $(a, b, c) = (4, 6, 10)$, giving $(2+3+23+5)(4+6+10) = 33 \cdot 20 = 660.$

13. In triangle ABC, point D is on $\overline{BC}$ with $CD = 2$ and $DB = 5$, point E is on $\overline{AC}$ with $CE = 1$ and $EA = 3$, $AB = 8$, and $\overline{AD}$ and $\overline{BE}$ intersect at P. Points Q and R lie on $\overline{AB}$ so that $\overline{PQ}$ is parallel to $\overline{CA}$ and $\overline{PR}$ is parallel to $\overline{CB}$. It is given that the ratio of the area of triangle PQR to the area of triangle ABC is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Assign masses 5 at A, 6 at B, and 15 at C. Then E balances $\overline{AC}$ ($5 \cdot 3 = 15 \cdot 1$) and D balances $\overline{BC}$ ($6 \cdot 5 = 15 \cdot 2$), so the cevians $\overline{AD}$ and $\overline{BE}$ meet at the center of mass P, of total mass 26. Extending $\overline{CP}$ to meet $\overline{AB}$ at F, the mass at F is $5 + 6 = 11$, so on segment CF we get $CP : PF = 11 : 15$, that is, $\frac{FP}{FC} = \frac{15}{26}$.

The homothety centered at F with ratio $\frac{15}{26}$ sends C to P and maps line AB to itself; it carries line CA to the parallel line through P — which is line PQ — and line CB to line PR. Hence it maps triangle CAB onto triangle PQR, and

$$\frac{[PQR]}{[ABC]} = \left(\frac{15}{26}\right)^2 = \frac{225}{676}.$$

Since $\gcd(225, 676) = 1$, the answer is $m + n = 225 + 676 = 901$.

14. The perimeter of triangle APM is 152, and angle PAM is a right angle. A circle of radius 19 with center O on $\overline{AP}$ is drawn so that it is tangent to $\overline{AM}$ and $\overline{PM}$. Given that $OP = m/n$, where m and n are relatively prime positive integers, find $m + n$.

Solution:

Let T be the point where the circle touches $\overline{PM}$. Since $\overline{AM} \perp \overline{AP}$ and O lies on $\overline{AP}$ at distance 19 from line AM, the circle is tangent to $\overline{AM}$ at A itself, so the two tangents from M give $MT = MA$. Right triangles POT and PMA (right angles at T and A) share angle P, so they are similar with ratio $\frac{OT}{MA} = \frac{19}{MA}$.

The small triangle's perimeter is $PO + OT + TP = (PA - 19) + 19 + TP = PA + PT$, and since $MT = MA$,

$$PA + PT = PA + PM - MT = 152 - 2\,MA.$$

Perimeters of similar triangles are in the ratio of similarity, so $\frac{19}{MA} = \frac{152-2\,MA}{152}$, which simplifies to $MA^2 - 76\,MA + 1444 = (MA - 38)^2 = 0$. Thus $MA = 38$.

The ratio of similarity is then $\frac{19}{38} = \frac{1}{2}$, so $PO = \frac{1}{2}PM$. From the perimeter, $PA + PM = 152 - 38 = 114$, and $PA = PO + 19$, so $\frac{1}{2}PM + 19 + PM = 114$, giving $PM = \frac{190}{3}$ and $OP = \frac{95}{3}$. Hence $m + n = 95 + 3 = 98$.

15. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points, one of which is $(9,6)$, and the product of their radii is 68. The x-axis and the line $y = mx$, where $m > 0$, are tangent to both circles. It is given that m can be written in the form $a\sqrt{b}/c$, where a, b, and c are positive integers, b is not divisible by the square of any prime, and a and c are relatively prime. Find $a+b+c$.

Solution:

Both circles are tangent to the x-axis and to $y = mx$, so both centers lie on the bisector of the first-quadrant angle between those lines. If the bisector makes angle α with the x-axis, then $m = \tan 2\alpha$, and each center has the form $(x_i,\ x_i \tan\alpha)$ with radius $r_i = x_i \tan\alpha$ (its distance to the x-axis).

Since $(9,6)$ lies on each circle, $(9 - x_i)^2 + (6 - x_i\tan\alpha)^2 = x_i^2 \tan^2\alpha$, which expands to

$$x_i^2 - (18 + 12\tan\alpha)\, x_i + 117 = 0.$$

Both x_1 and x_2 satisfy this one quadratic, so by Vieta's formulas $x_1 x_2 = 117$. Then $r_1 r_2 = x_1 x_2 \tan^2\alpha = 117\tan^2\alpha = 68$, so $\tan^2\alpha = \frac{68}{117}$ and $\tan\alpha = \frac{2\sqrt{17}}{3\sqrt{13}}$.

Finally,

$$m = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2\tan\alpha}{49/117} = \frac{234}{49}\cdot\frac{2\sqrt{17}}{3\sqrt{13}} = \frac{156\sqrt{17}}{49\sqrt{13}} = \frac{12\sqrt{221}}{49},$$

so $a + b + c = 12 + 221 + 49 = 282.$

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