2002 AIME II

Time limit: 180 minutes

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1. Given that

(1) x and y are both integers between 100 and $999,$ inclusive;

(2) y is the number formed by reversing the digits of x; and

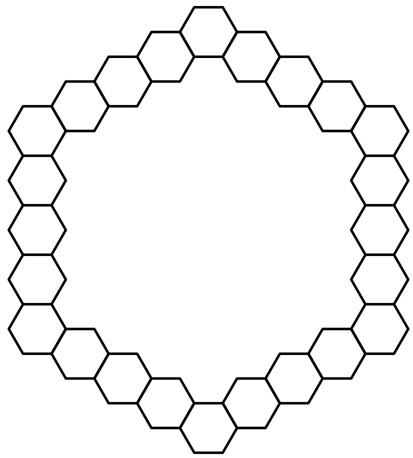
(3) $z = |x - y|$.

How many distinct values of z are possible?

2. Three of the vertices of a cube are $P = (7, 12, 10)$, $Q = (8, 8, 1)$, and $R = (11, 3, 9)$. What is the surface area of the cube?

3. It is given that $\log_6 a + \log_6 b + \log_6 c = 6,$ where $a, b,$ and c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

4. Patio blocks that are regular hexagons 1 unit on a side are used to outline a garden by placing the blocks edge to edge with n on each side. The diagram indicates the path of blocks around the garden when $n = 5$.



If $n = 202$, then the area of the garden enclosed by the path, not including the path itself, is $m\left(\sqrt{3}/2\right)$ square units, where m is a positive integer. Find the remainder when m is divided by 1000.

5. Find the sum of all positive integers $a = 2^n 3^m$, where n and m are non-negative integers, for which a^6 is not a divisor of 6^a.

6. Find the integer that is closest to

$$1000 \sum_{n=3}^{10000} \frac{1}{n^2 - 4}.$$

7. It is known that, for all positive integers k,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Find the smallest positive integer k such that $1^2 + 2^2 + 3^2 + \cdots + k^2$ is a multiple of 200.

8. Find the least positive integer k for which the equation $\left\lfloor \frac{2002}{n} \right\rfloor = k$ has no integer solutions for n. (The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x.)

9. Let $\mathcal{S}$ be the set $\{1, 2, 3, \ldots, 10\}$. Let n be the number of sets of two non-empty disjoint subsets of $\mathcal{S}$. (*Disjoint sets* are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000.

10. While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of x for which the sine of x degrees is the same as the sine of x radians are $\frac{m\pi}{n-\pi}$ and $\frac{p\pi}{q+\pi}$, where $m, n, p,$ and q are positive integers. Find $m + n + p + q$.

11. Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is $\frac{1}{8}$, and the second term of both series can be written in the form $\frac{\sqrt{m}-n}{p}$, where $m, n,$ and p are positive integers and m is not divisible by the square of any prime. Find $100m + 10n + p$.

12. A basketball player has a constant probability of $.4$ of making any given shot, independent of previous shots. Let a_n be the ratio of shots made to shots attempted after n shots. The probability that $a_{10} = .4$ and $a_n \leq .4$ for all n such that $1 \leq n \leq 9$ is given to be $p^a q^b r / \left(s^c\right)$, where $p, q, r,$ and s are primes, and $a, b,$ and c are positive integers. Find $(p + q + r + s)(a + b + c)$.

13. In triangle ABC, point D is on $\overline{BC}$ with $CD = 2$ and $DB = 5$, point E is on $\overline{AC}$ with $CE = 1$ and $EA = 3$, $AB = 8$, and $\overline{AD}$ and $\overline{BE}$ intersect at P. Points Q and R lie on $\overline{AB}$ so that $\overline{PQ}$ is parallel to $\overline{CA}$ and $\overline{PR}$ is parallel to $\overline{CB}$. It is given that the ratio of the area of triangle PQR to the area of triangle ABC is m/n, where m and n are relatively prime positive integers. Find $m + n$.

14. The perimeter of triangle APM is 152, and angle PAM is a right angle. A circle of radius 19 with center O on $\overline{AP}$ is drawn so that it is tangent to $\overline{AM}$ and $\overline{PM}$. Given that $OP = m/n$, where m and n are relatively prime positive integers, find $m + n$.

15. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points, one of which is $(9, 6)$, and the product of their radii is 68. The x-axis and the line $y = mx$, where $m > 0$, are tangent to both circles. It is given that m can be written in the form $a\sqrt{b}/c$, where a, b, and c are positive integers, b is not divisible by the square of any prime, and a and c are relatively prime. Find $a + b + c$.

Solutions: https://live.poshenloh.com/past-contests/aime/2002II/solutions