2002 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/aime/2002I/solutions

1. Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

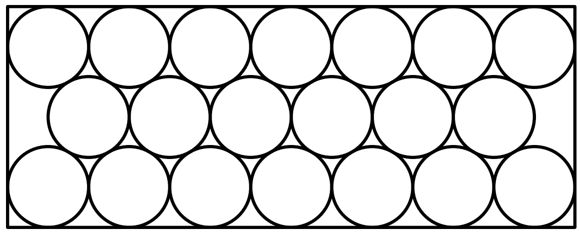
A three-letter arrangement is a palindrome exactly when the third letter matches the first, so the probability of a letter palindrome is $\frac{1}{26}$. Similarly, the probability of a digit palindrome is $\frac{1}{10}$, and the two events are independent.

By inclusion-exclusion, the probability of at least one palindrome is

$$\frac{1}{26}+\frac{1}{10}-\frac{1}{26}\cdot\frac{1}{10}=\frac{10+26-1}{260}=\frac{35}{260}=\frac{7}{52}.$$

Thus $m+n=7+52=59.$

2. The diagram shows twenty congruent circles arranged in three rows and enclosed in a rectangle. The circles are tangent to one another and to the sides of the rectangle as shown in the diagram. The ratio of the longer dimension of the rectangle to the shorter dimension can be written as $\frac{1}{2}\left(\sqrt{p}-q\right)$, where p and q are positive integers. Find $p+q$.



Solution:

Let r be the common radius. The longer side holds a row of seven circles, so it equals $14r$. The centers of three mutually tangent circles in adjacent rows form an equilateral triangle with side $2r$, whose height is $r\sqrt{3}$, so the two gaps between rows of centers contribute $2r\sqrt{3}$, and the shorter side is $r+2r\sqrt{3}+r=2r+2r\sqrt{3}$.

The ratio is

$$\frac{14r}{2r\left(1+\sqrt{3}\right)}=\frac{7}{1+\sqrt{3}}=\frac{7\left(\sqrt{3}-1\right)}{2}=\frac{1}{2}\left(\sqrt{147}-7\right),$$

so $p=147, q=7$, and $p+q=154$.

3. Jane is 25 years old. Dick is older than Jane. In n years, where n is a positive integer, Dick's age and Jane's age will both be two-digit numbers and will have the property that Jane's age is obtained by interchanging the digits of Dick's age. Let d be Dick's present age. How many ordered pairs of positive integers (d, n) are possible?

Solution:

In n years Jane's age is $25 + n$, and Dick's age is its digit reversal. If Jane's future age is $10a + b$, Dick's is $10b + a$, which is larger exactly when $b > a$. Conversely, every two-digit value of $25 + n$ with tens digit less than units digit yields exactly one valid pair: $n = 10a + b - 25$ and $d = 10b + a - n = 25 + 9(b - a) > 25$, so Dick is indeed older than Jane now.

So we count two-digit numbers that are at least 26 and have tens digit less than units digit: 4 starting with 2 (namely 26 through 29), then $6, 5, 4, 3, 2, 1$ starting with 3 through 8. The total is $4 + 6 + 5 + 4 + 3 + 2 + 1 = 25$.

4. Consider the sequence defined by $a_k = \frac{1}{k^2+k}$ for $k \geq 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = \frac{1}{29}$, for positive integers m and n with $m < n$, find $m + n$.

Solution:

Since $\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, the sum telescopes:

$$a_m + a_{m+1} + \cdots + a_{n-1} = \frac{1}{m} - \frac{1}{n} = \frac{1}{29}.$$

Multiplying through by $29mn$ gives $29n - 29m = mn$, which rearranges to $(29 - m)(29 + n) = 29^2$. Since 29 is prime and $29 + n > 29$, the only factorization with m a positive integer is $29 - m = 1$ and $29 + n = 841$, so $m = 28$ and $n = 812$.

Therefore $m + n = 28 + 812 = 840$.

5. Let $A_1, A_2, A_3, \ldots, A_{12}$ be the vertices of a regular dodecagon. How many distinct squares in the plane of the dodecagon have at least two vertices in the set $\{A_1, A_2, A_3, \ldots, A_{12}\}$?

Solution:

Each of the $\binom{12}{2} = 66$ pairs of vertices determines exactly three squares: two having the pair as a side (one on each side of the segment) and one having it as a diagonal. That counts $3 \cdot 66 = 198$ squares.

A square is overcounted only if it has more than two vertices among the A_i. If three vertices of a square lie on the circumcircle, the square's own circumcircle shares three points with it and hence coincides with it, and an inscribed square's vertices are spaced 90° apart — three steps of the dodecagon — so the fourth vertex is also an A_i. The fully inscribed squares are exactly $A_1A_4A_7A_{10}$, $A_2A_5A_8A_{11}$, and $A_3A_6A_9A_{12}$, and each is generated by all $\binom{4}{2} = 6$ of its vertex pairs, so each is counted 6 times instead of once.

The number of distinct squares is $198 - 3 \cdot 5 = 183$.

6. The solutions to the system of equations

$$\log_{225} x + \log_{64} y = 4$$

$$\log_x 225 - \log_y 64 = 1$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1 y_1 x_2 y_2)$.

Solution:

Let $p = \log_{225} x$ and $q = \log_{64} y$, so $\log_x 225 = \frac{1}{p}$ and $\log_y 64 = \frac{1}{q}$. The system becomes $p + q = 4$ and $\frac{1}{p} - \frac{1}{q} = 1$. Substituting $q = 4 - p$ into the second equation and clearing denominators gives $4 - 2p = p(4 - p)$, that is, $p^2 - 6p + 4 = 0$.

The two solutions of the system correspond to the two roots of this quadratic, so by Vieta's formulas $p_1 + p_2 = 6$, and then $q_1 + q_2 = 8 - 6 = 2$. Hence

$$x_1 y_1 x_2 y_2 = 225^{p_1 + p_2} \cdot 64^{q_1 + q_2} = 225^6 \cdot 64^2 = 15^{12} \cdot 2^{12} = 30^{12},$$

so $\log_{30}(x_1 y_1 x_2 y_2) = 12.$

7. The Binomial Expansion is valid for exponents that are not integers. That is, for all real numbers x, y, and r with $|x| > |y|$,

$$(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \cdots$$

What are the first three digits to the right of the decimal point in the decimal representation of $\left(10^{2002}+1\right)^{10/7}$?

Solution:

Apply the expansion with $x = 10^{2002}$, $y = 1$, and $r = \frac{10}{7}$:

$$\left(10^{2002}+1\right)^{10/7} = 10^{2860} + \frac{10}{7} \cdot 10^{858} + \frac{\frac{10}{7} \cdot \frac{3}{7}}{2} \cdot 10^{-1144} + \cdots .$$

The first term is an integer, and the third and later terms are far smaller than 10^{-1000}, too small to affect the leading decimal digits. So those digits come from the fractional part of $\frac{10}{7} \cdot 10^{858} = \frac{10^{859}}{7}$.

That fractional part is $\frac{10^{859} \bmod 7}{7}$. Since $10^6 \equiv 1 \pmod 7$ and $859 \equiv 1 \pmod 6$, we get $10^{859} \equiv 10 \equiv 3 \pmod 7$, so the fractional part is $\frac{3}{7} = 0.428571\ldots$

The first three digits to the right of the decimal point are 428.

8. Find the smallest integer k for which the conditions

- $a_1, a_2, a_3, \ldots$ is a nondecreasing sequence of positive integers
- $a_n = a_{n-1} + a_{n-2}$ for all $n > 2$
- $a_9 = k$

are satisfied by more than one sequence.

Solution:

Iterating the recurrence gives $a_9 = 13a_1 + 21a_2$, and the sequence is nondecreasing exactly when $0 < a_1 \leq a_2$ (all later terms then take care of themselves). So we need the smallest k for which $13x + 21y = k$ has two solutions with $0 < x \leq y$.

Suppose $13x + 21y = 13u + 21v$ with $x < u$. Then $13(u - x) = 21(y - v)$, so $u - x$ is a positive multiple of 21. Hence $u \geq x + 21 \geq 22$, and since $u \leq v$, also $v \geq 22$, giving $k = 13u + 21v \geq 34 \cdot 22 = 748$.

Conversely $k = 748$ works: $(x, y) = (1, 35)$ and $(22, 22)$ give the sequences $1, 35, 36, 71, 107, 178, 285, 463, 748$ and $22, 22, 44, 66, 110, 176, 286, 462, 748$. The answer is $k = 748$.

9. Harold, Tanya, and Ulysses paint a very long picket fence.

- Harold starts with the first picket and paints every hth picket;
- Tanya starts with the second picket and paints every tth picket; and
- Ulysses starts with the third picket and paints every uth picket.

Call the positive integer $100h + 10t + u$ *paintable* when the triple (h, t, u) of positive integers results in every picket being painted exactly once. Find the sum of all the paintable integers.

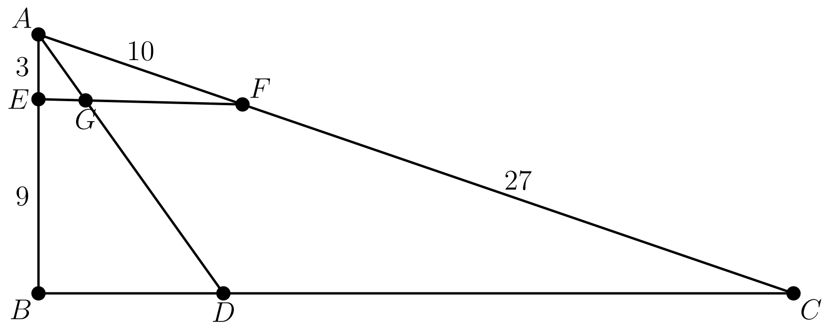
Solution:

The three progressions $\{1, 1+h, \ldots\}, \{2, 2+t, \ldots\}, \{3, 3+u, \ldots\}$ must partition the positive integers. If $h = 2,$ Harold paints picket $3,$ which Ulysses also paints, so $h \geq 3.$ If $h \geq 5,$ consider picket $4:$ Harold's next picket is $1 + h \geq 6,$ and Ulysses cannot paint it (that would need $u = 1,$ repainting everything from 3 on), so Tanya must, forcing $t = 2.$ Then picket 5 is unpainted unless $u = 2,$ but then Tanya and Ulysses together cover every picket from 2 on, and Harold's picket $1 + h$ is painted twice. So $h = 3$ or $h = 4.$

If $h = 3,$ Harold paints $1, 4, 7, \ldots.$ Ulysses cannot paint picket 5 (then $u = 2$ and he would repaint 7), so Tanya does: $t = 3,$ covering $2, 5, 8, \ldots.$ What remains is exactly $3, 6, 9, \ldots,$ so $u = 3,$ giving $333.$ If $h = 4,$ Harold paints $1, 5, 9, \ldots$; picket 4 again forces $t = 2,$ and the leftover pickets $3, 7, 11, \ldots$ force $u = 4,$ giving $424.$

The sum of the paintable integers is $333 + 424 = 757.$

10. In the diagram below, angle ABC is a right angle. Point D is on $\overline{BC}$, and $\overline{AD}$ bisects angle CAB. Points E and F are on $\overline{AB}$ and $\overline{AC}$, respectively, so that $AE = 3$ and $AF = 10$. Given that $EB = 9$ and $FC = 27$, find the integer closest to the area of quadrilateral $DCFG$.



Solution:

Here $AB = 3 + 9 = 12$, $AC = 10 + 27 = 37$, and angle B is right, so $BC = \sqrt{37^2 - 12^2} = 35$ and $[ABC] = \frac{1}{2} \cdot 12 \cdot 35 = 210$. The quadrilateral is triangle ADC with triangle AGF removed, where G is the intersection of $\overline{AD}$ and $\overline{EF}$.

By the angle bisector theorem in triangle ABC, $BD : DC = AB : AC = 12 : 37$, so $[ADC] = \frac{37}{49} \cdot 210 = \frac{1110}{7}$. In triangle AEF, ray AG bisects the same angle, so $EG : GF = AE : AF = 3 : 10$ and $[AGF] = \frac{10}{13}[AEF]$. Also $[AEF] = \frac{AE}{AB} \cdot \frac{AF}{AC}[ABC] = \frac{3}{12} \cdot \frac{10}{37} \cdot 210 = \frac{525}{37}$.

Therefore

$$[DCFG] = \frac{1110}{7} - \frac{10}{13} \cdot \frac{525}{37} = \frac{1110}{7} - \frac{5250}{481} \approx 158.57 - 10.92 = 147.66,$$

and the closest integer is 148.

11. Let $ABCD$ and $BCFG$ be two faces of a cube with $AB = 12.$ A beam of light emanates from vertex A and reflects off face $BCFG$ at point P, which is 7 units from $\overline{BG}$ and 5 units from $\overline{BC}$. The beam continues to be reflected off the faces of the cube. The length of the light path from the time it leaves point A until it next reaches a vertex of the cube is given by $m\sqrt{n}$, where m and n are integers and n is not divisible by the square of any prime. Find $m + n$.

Solution:

Place $A = (0, 0, 0)$ with the cube $[0, 12]^3$ and $P = (12, 7, 5)$ on the face $x = 12$. Reflecting the cube across the relevant face at each bounce straightens the reflected path into the straight ray from A through P : each crossing of a plane $x = 12k, y = 12k$, or $z = 12k$ corresponds to a bounce, and the beam reaches a vertex of the cube exactly when all three coordinates are simultaneously multiples of 12.

The ray consists of the points $(12t, 7t, 5t)$. Since 7 and 5 are relatively prime to 12, the coordinates $7t$ and $5t$ are first divisible by 12 when $t = 12$, at the point $(144, 84, 60)$. The path length equals the straight-line distance

$$\sqrt{144^2 + 84^2 + 60^2} = 12\sqrt{12^2 + 7^2 + 5^2} = 12\sqrt{218}.$$

Since $218 = 2 \cdot 109$ is squarefree, $m + n = 12 + 218 = 230.$

12. Let $F(z) = \frac{z+i}{z-i}$ for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that $z_0 = \frac{1}{137} + i$ and $z_{2002} = a + bi$, where a and b are real numbers, find $a + b$.

Solution:

Composing the map with itself,

$$F(F(z)) = \frac{\frac{z+i}{z-i} + i}{\frac{z+i}{z-i} - i} = \frac{(z+i) + i(z-i)}{(z+i) - i(z-i)} = \frac{(1+i)(z+1)}{(1-i)(z-1)} = i\,\frac{z+1}{z-1},$$

and applying F once more gives

$$F(F(F(z))) = \frac{i\,\frac{z+1}{z-1} + i}{i\,\frac{z+1}{z-1} - i} = \frac{(z+1) + (z-1)}{(z+1) - (z-1)} = z,$$

so the sequence $z_0, z_1, z_2, \ldots$ is periodic with period 3.

Since $2002 = 3 \cdot 667 + 1$, we have $z_{2002} = z_1 = F(z_0) = \frac{z_0+i}{z_0-i} = \frac{\frac{1}{137}+2i}{\frac{1}{137}} = 1 + 274i$. Thus $a + b = 1 + 274 = 275$.

13. In triangle ABC, the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Solution:

Since E is the midpoint of $\overline{AB}$, $AE = EB = 12$. Let P be the centroid, which trisects the medians: $AP = \frac{2}{3} \cdot 18 = 12$ and $PE = \frac{1}{3} \cdot 27 = 9$. By the power of the point E with respect to the circumcircle, $EF \cdot EC = EA \cdot EB = 144$, so $EF = \frac{144}{27} = \frac{16}{3}$.

Triangle AEP is isosceles with $AE = AP = 12$ and base $PE = 9$, so the altitude from A to $\overline{PE}$ is $\sqrt{144 - \frac{81}{4}} = \frac{3\sqrt{55}}{2}$, giving $[AEP] = \frac{1}{2} \cdot 9 \cdot \frac{3\sqrt{55}}{2} = \frac{27\sqrt{55}}{4}$. Since F and P both lie on line CE, triangles AEF and AEP share the apex A and have collinear bases, so

$$[AEF] = \frac{EF}{EP}\,[AEP] = \frac{16/3}{9} \cdot \frac{27\sqrt{55}}{4} = 4\sqrt{55}.$$

Finally, since E is the midpoint of $\overline{AB}$, $[AFB] = 2\,[AFE] = 8\sqrt{55}$, and $m + n = 8 + 55 = 63$.

14. A set $\mathcal{S}$ of distinct positive integers has the following property: for every integer x in $\mathcal{S}$, the arithmetic mean of the set of values obtained by deleting x from $\mathcal{S}$ is an integer. Given that 1 belongs to $\mathcal{S}$ and that 2002 is the largest element of $\mathcal{S}$, what is the greatest number of elements that $\mathcal{S}$ can have?

Solution:

Let $\mathcal{S}$ have n elements with sum S. The condition says $\frac{S-x}{n-1}$ is an integer for every $x \in \mathcal{S}$, which means every element is congruent to S modulo $n-1$. In particular all elements are congruent to each other, and since $1 \in \mathcal{S}$, every element is 1 more than a multiple of $n-1$.

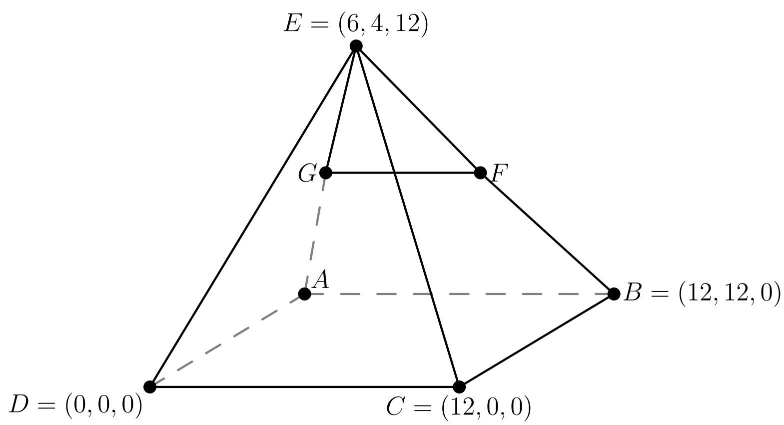
Then $2002 \equiv 1 \pmod{n-1}$, so $n-1$ divides $2001 = 3 \cdot 23 \cdot 29$. Moreover the n distinct elements run from 1 up to 2002 in steps that are multiples of $n-1$, so $2002 \geq 1 + (n-1)^2$, forcing $n - 1 \leq 44$. The largest divisor of 2001 that is at most 44 is 29, so $n \leq 30$.

Thirty is attainable: take the 29 numbers $1, 30, 59, \ldots, 813$ together with 2002. All are $\equiv 1 \pmod{29}$, and the sum of all 30 is $\equiv 30 \equiv 1 \pmod{29}$, so every deleted mean is an integer. The answer is 30.

15. Polyhedron $ABCDEFG$ has six faces. Face $ABCD$ is a square with $AB = 12$; face $ABFG$ is a trapezoid with $\overline{AB}$ parallel to $\overline{GF}$, $BF = AG = 8$, and $GF = 6$; and face CDE has $CE = DE = 14$. The other three faces are $ADEG$, $BCEF$, and EFG. The distance from E to face $ABCD$ is 12. Given that $EG^2 = p - q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.

Solution:

Place $D = (0,0,0)$, $C = (12,0,0)$, $B = (12,12,0)$, $A = (0,12,0)$, and $E = (x, y, 12)$, using the given distance from E to face $ABCD$. From $CE = DE$ we get $x = 6$, and then $DE = 14$ gives $36 + y^2 + 144 = 196$, so $y = 4$ and $E = (6,4,12)$.



In trapezoid $ABFG$, $\overline{GF}$ is parallel to $\overline{AB}$ with $GF = 6$ and $AG = BF$, so G and F are symmetric about the plane $x = 6 : G = (3, y_2, z_2)$ and $F = (9, y_2, z_2)$. Face $ADEG$ is planar, and the plane through A, D, E contains the entire y-axis direction (both A and D have $x = z = 0$), so it is the plane $z = 2x$, which indeed contains E. Hence $z_2 = 6$. Now $AG = 8$ gives $3^2 + (y_2 - 12)^2 + 6^2 = 64$, so $y_2 = 12 \pm \sqrt{19}$.

Then

$$EG^2 = 3^2 + (y_2 - 4)^2 + 6^2 = 45 + \left(8 \pm \sqrt{19}\right)^2 = 128 \pm 16\sqrt{19},$$

and the stated form $p - q\sqrt{r}$ corresponds to $128 - 16\sqrt{19}$. Thus $p + q + r = 128 + 16 + 19 = 163$.

Problems: https://live.poshenloh.com/past-contests/aime/2002I