2001 AIME II Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/aime/2001II/solutions

1. Let N be the largest positive integer with the following property: reading from left to right, each pair of consecutive digits of N forms a perfect square. What are the leftmost three digits of N?

Solution:

Each pair of consecutive digits must be one of the two-digit squares $16, 25, 36, 49, 64, 81$. So each digit determines its successor uniquely if one exists: $1 \to 6, 2 \to 5, 3 \to 6, 4 \to 9, 6 \to 4, 8 \to 1$, while 5 and 9 end the number.

Following these chains from each possible starting digit, the longest strings are 25, 3649, and 81649. The five-digit chain $8 \to 1 \to 6 \to 4 \to 9$ beats everything else, so $N = 81649$, whose leftmost three digits are 816.

2. Each of the 2001 students at a high school studies either Spanish or French, and some study both. The number who study Spanish is between 80 percent and 85 percent of the school population, and the number who study French is between 30 percent and 40 percent. Let m be the smallest number of students who could study both languages, and let M be the largest number of students who could study both languages. Find $M - m$.

Solution:

Let s and f be the numbers of students studying Spanish and French. Since every student studies at least one language, the number studying both is $s + f - 2001$. The bounds $1600.8 < s < 1700.85$ force $1601 \leq s \leq 1700$, and $600.3 < f < 800.4$ force $601 \leq f \leq 800$.

The overlap is smallest when $s + f$ is smallest, giving $m = 1601 + 601 - 2001 = 201$, and largest when $s + f$ is largest, giving $M = 1700 + 800 - 2001 = 499$. Both extremes are achievable, so $M - m = 499 - 201 = 298$.

3. Given that $x_1 = 211, x_2 = 375, x_3 = 420, x_4 = 523$, and

$$x_n = x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4} \quad \text{when } n \geq 5,$$

find the value of $x_{531} + x_{753} + x_{975}$.

Solution:

For $n \geq 6$, substitute the recurrence for x_{n-1} :

$$x_n = (x_{n-2} - x_{n-3} + x_{n-4} - x_{n-5}) - x_{n-2} + x_{n-3} - x_{n-4} = -x_{n-5}.$$

Hence $x_{n+10} = -x_{n+5} = x_n$, so the sequence has period 10.

Since $531, 753$, and 975 leave remainders $1, 3$, and 5 upon division by 10, we get $x_{531} = x_1 = 211, x_{753} = x_3 = 420$, and $x_{975} = x_5 = x_4 - x_3 + x_2 - x_1 = 523 - 420 + 375 - 211 = 267$.

The sum is $211 + 420 + 267 = 898$.

4. Let $R = (8, 6)$. The lines whose equations are $8y = 15x$ and $10y = 3x$ contain points P and Q, respectively, such that R is the midpoint of $\overline{PQ}$. The length PQ equals $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Points on the two lines can be written $P = (8t, 15t)$ and $Q = (10u, 3u)$. Since $R = (8, 6)$ is the midpoint of $\overline{PQ}$,

$$8t + 10u = 16 \qquad \text{and} \qquad 15t + 3u = 12.$$

The second equation gives $u = 4 - 5t$; substituting into the first, $8t + 40 - 50t = 16$, so $t = \frac{4}{7}$ and $u = \frac{8}{7}$. Thus $P = \left(\frac{32}{7}, \frac{60}{7}\right)$ and $Q = \left(\frac{80}{7}, \frac{24}{7}\right)$.

Then $PQ = \sqrt{\left(\frac{48}{7}\right)^2 + \left(\frac{36}{7}\right)^2} = \frac{12}{7}\sqrt{4^2 + 3^2} = \frac{60}{7}$, so $m + n = 60 + 7 = 67$.

5. A set of positive numbers has the *triangle property* if it has three distinct elements that are the lengths of the sides of a triangle whose area is positive. Consider sets $\{4, 5, 6, \ldots, n\}$ of consecutive positive integers, all of whose ten-element subsets have the triangle property. What is the largest possible value of n?

Solution:

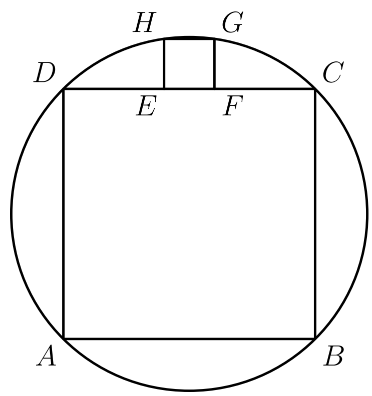
Suppose a ten-element set $\{a_1 < a_2 < \cdots < a_{10}\}$ has no triangle. Then every three elements fail the strict triangle inequality; in particular $a_{i+2} \geq a_{i+1} + a_i$ for each i. Starting from $a_1 \geq 4$ and $a_2 \geq 5$, this forces $a_3 \geq 9, a_4 \geq 14, a_5 \geq 23, a_6 \geq 37,$ $a_7 \geq 60, a_8 \geq 97, a_9 \geq 157,$ and $a_{10} \geq 254.$

So if $n \leq 253,$ no ten-element subset of $\{4, 5, \ldots, n\}$ can avoid triangles, since its largest element would have to be at least $254.$ Conversely, taking equality throughout, the subset $\{4, 5, 9, 14, 23, 37, 60, 97, 157, 254\}$ of $\{4, 5, \ldots, 254\}$ has no triangle.

Therefore the largest possible value is $n = 253.$

6. Square $ABCD$ is inscribed in a circle. Square $EFGH$ has vertices E and F on $\overline{CD}$ and vertices G and H on the circle. The ratio of the area of square $EFGH$ to the area of square $ABCD$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers and $m < n$. Find $10n + m$.

Solution:



Center the circle at the origin and let $ABCD$ have side s, so the circle is $x^2 + y^2 = \frac{s^2}{2}$ and side $\overline{CD}$ lies on the line $y = \frac{s}{2}$. The small square sits on $\overline{CD}$, outside $ABCD$: if its side is t, then by symmetry $G = \left(\frac{t}{2}, \frac{s}{2} + t\right)$, which must lie on the circle.

Substituting, $\frac{t^2}{4} + \left(\frac{s}{2} + t\right)^2 = \frac{s^2}{2}$, which expands to $5t^2 + 4st - s^2 = 0$, or $(5t - s)(t + s) = 0$. Since $t > 0$, we get $t = \frac{s}{5}$.

The ratio of areas is $\frac{t^2}{s^2} = \frac{1}{25}$, so $m = 1, n = 25$, and $10n + m = 251$.

7. Let $\triangle PQR$ be a right triangle with $PQ = 90, PR = 120$, and $QR = 150$. Let C_1 be the inscribed circle. Construct $\overline{ST}$, with S on $\overline{PR}$ and T on $\overline{QR}$, such that $\overline{ST}$ is perpendicular to $\overline{PR}$ and tangent to C_1. Construct $\overline{UV}$ with U on $\overline{PQ}$ and V on $\overline{QR}$ such that $\overline{UV}$ is perpendicular to $\overline{PQ}$ and tangent to C_1. Let C_2 be the inscribed circle of $\triangle RST$ and C_3 the inscribed circle of $\triangle QUV$. The distance between the centers of C_2 and C_3 can be written as $\sqrt{10n}$. What is n?

Solution:

The right angle is at P, so place $P = (0, 0), Q = (0, 90), R = (120, 0)$. The inradius of a right triangle is half the sum of the legs minus the hypotenuse: $r_1 = \frac{90+120-150}{2} = 30$, so C_1 has center $(30, 30)$. The tangent line to C_1 perpendicular to $\overline{PR}$ (on the side toward R) is $x = 60$, and the tangent perpendicular to $\overline{PQ}$ (toward Q) is $y = 60$.

Triangle RST is similar to triangle RPQ with ratio $\frac{RS}{RP} = \frac{60}{120} = \frac{1}{2}$, so its inradius is 15 and its incircle C_2 is centered at $(60 + 15, 15) = (75, 15)$. Triangle QUV is similar to triangle QPR with ratio $\frac{QU}{QP} = \frac{30}{90} = \frac{1}{3}$, so its inradius is 10 and C_3 is centered at $(10, 60 + 10) = (10, 70)$.

The squared distance is $65^2 + 55^2 = 4225 + 3025 = 7250 = 10 \cdot 725$, so $n = 725$.

8. A certain function f has the properties that $f(3x) = 3f(x)$ for all positive real values of x, and that $f(x) = 1 - |x - 2|$ for $1 \leq x \leq 3$. Find the smallest x for which $f(x) = f(2001)$.

Solution:

Applying $f(3x) = 3f(x)$ six times gives $f(2001) = 3^6 f\left(\frac{2001}{729}\right)$, and $\frac{2001}{729}$ lies in $[1, 3]$, so

$$f(2001) = 729\left(1 - \left|\frac{2001}{729} - 2\right|\right) = 729 - |2001 - 1458| = 729 - 543 = 186.$$

For $x \in [3^k, 3^{k+1}]$, we have $f(x) = 3^k f\left(\frac{x}{3^k}\right) = 3^k\left(1 - \left|\frac{x}{3^k} - 2\right|\right)$, a tent whose maximum value is 3^k. To achieve 186 we need $3^k \geq 186$, so $k \geq 5$, and the smallest solutions lie in $[243, 729]$, where $f(x) = 243 - |x - 486|$.

Setting $243 - |x - 486| = 186$ gives $|x - 486| = 57$, so $x = 429$ or $x = 543$. The smallest x is 429.

9. Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. The probability of obtaining a grid that does not have a 2-by-2 red square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Compute the probability that the grid *does* contain an all-red 2-by-2 block by inclusion-exclusion over the four possible positions. One block forces 4 cells; two blocks sharing an edge force 6 cells (4 such pairs), while the two diagonal pairs force 7; any three blocks force 8 cells, and all four force all 9.

Each configuration of forced red cells has probability $\left(\frac{1}{2}\right)^{\text{cells}}$, so the probability of at least one red block is

$$4 \cdot \frac{1}{16} - \left(4 \cdot \frac{1}{64} + 2 \cdot \frac{1}{128}\right) + 4 \cdot \frac{1}{256} - \frac{1}{512} = \frac{128 - 40 + 8 - 1}{512} = \frac{95}{512}.$$

The desired probability is $1 - \frac{95}{512} = \frac{417}{512}$, and $417 = 3 \cdot 139$ is coprime to 512, so $m + n = 417 + 512 = 929.$

10. How many positive integer multiples of 1001 can be expressed in the form $10^j - 10^i$, where i and j are integers and $0 \le i < j \le 99$?

Solution:

Factor $10^j - 10^i = 10^i(10^{j-i} - 1)$. Since $1001 = 7 \cdot 11 \cdot 13$ is coprime to 10^i, we need $1001 \mid 10^{j-i} - 1$. The multiplicative order of 10 is 6 modulo 7, 2 modulo 11, and 6 modulo 13, so $10^k \equiv 1 \pmod{1001}$ exactly when k is a multiple of 6. Distinct pairs (i, j) give distinct values, so we just count the pairs.

For $j - i = 6d$ with $1 \le d \le 16$, the index i can be $0, 1, \ldots, 99 - 6d$, giving $100 - 6d$ choices. The total is

$$\sum_{d=1}^{16}(100 - 6d) = 94 + 88 + \cdots + 4 = \frac{(94 + 4) \cdot 16}{2} = 784.$$

11. Club Truncator is in a soccer league with six other teams, each of which it plays once. In any of its 6 matches, the probabilities that Club Truncator will win, lose, or tie are each $\frac{1}{3}$. The probability that Club Truncator will finish the season with more wins than losses is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Swapping wins and losses is a probability-preserving symmetry, so the probability P of more wins than losses equals the probability of more losses than wins, giving $P = \frac{1-p_0}{2}$, where p_0 is the probability of equally many wins and losses.

An outcome with k wins, k losses, and $6 - 2k$ ties can be arranged in $\frac{6!}{k!\,k!\,(6-2k)!}$ ways: $1, 30, 90, 20$ for $k = 0, 1, 2, 3$, totaling 141. Each of the $3^6 = 729$ outcome sequences is equally likely, so $p_0 = \frac{141}{729} = \frac{47}{243}$.

Therefore $P = \frac{1}{2}\left(1 - \frac{47}{243}\right) = \frac{98}{243}$, and $m + n = 98 + 243 = 341.$

12. Given a triangle, its *midpoint triangle* is obtained by joining the midpoints of its sides. A sequence of polyhedra $\mathcal{P}_i$ is defined recursively as follows: $\mathcal{P}_0$ is a regular tetrahedron whose volume is 1. To obtain $\mathcal{P}_{i+1}$, replace the midpoint triangle of every face of $\mathcal{P}_i$ by an outward-pointing regular tetrahedron that has the midpoint triangle as a face. The volume of $\mathcal{P}_3$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Attaching a tetrahedron over the midpoint triangle of a face replaces that face by 6 equilateral triangles of half the side length: the 3 corner triangles plus 3 exposed faces of the new tetrahedron. So all faces of $\mathcal{P}_i$ are congruent, with side $\left(\frac{1}{2}\right)^i$ times the original, and $\mathcal{P}_i$ has $4 \cdot 6^i$ faces.

Passing from $\mathcal{P}_i$ to $\mathcal{P}_{i+1}$ glues one regular tetrahedron onto each face; each is similar to $\mathcal{P}_0$ with ratio $\left(\frac{1}{2}\right)^{i+1}$, hence has volume $\left(\frac{1}{8}\right)^{i+1}$. The volume added is

$$4 \cdot 6^i \left(\frac{1}{8}\right)^{i+1} = \frac{1}{2}\left(\frac{3}{4}\right)^i .$$

Therefore the volume of $\mathcal{P}_3$ is $1 + \frac{1}{2} + \frac{3}{8} + \frac{9}{32} = \frac{69}{32}$, and $m + n = 69 + 32 = 101.$

13. In quadrilateral $ABCD$, $\angle BAD \cong \angle ADC$ and $\angle ABD \cong \angle BCD$, $AB = 8$, $BD = 10$, and $BC = 6$. The length CD may be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Extend $\overline{AB}$ beyond B and $\overline{DC}$ beyond C to meet at P. Since $\angle PAD = \angle PDA$, triangle APD is isosceles with $PA = PD$. Also $\angle PBD = 180^\circ - \angle ABD$ and $\angle PCB = 180^\circ - \angle BCD$, so $\angle PBD = \angle PCB$.

Triangles PCB and PBD share angle P and have $\angle PCB = \angle PBD$, so they are similar, giving

$$\frac{PC}{PB} = \frac{PB}{PD} = \frac{CB}{BD} = \frac{3}{5}.$$

Since $PB = PA - 8 = PD - 8$, the middle ratio reads $\frac{PD-8}{PD} = \frac{3}{5}$, so $PD = 20$ and $PB = 12$. Then $PC = \frac{3}{5} \cdot 12 = \frac{36}{5}$.

Finally $CD = PD - PC = 20 - \frac{36}{5} = \frac{64}{5}$, which is in lowest terms, so $m + n = 64 + 5 = 69$.

14. There are $2n$ complex numbers that satisfy both $z^{28} - z^8 - 1 = 0$ and $|z| = 1$. These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \leq \theta_1 < \theta_2 < \ldots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \cdots + \theta_{2n}$.

Solution:

Write $\operatorname{cis}\theta = \cos\theta + i\sin\theta$. The equation says $z^8(z^{20} - 1) = 1$. Taking absolute values and using $|z| = 1$ gives $|z^{20} - 1| = 1$, so z^{20} is at distance 1 from both 0 and $1:$ it is $\operatorname{cis}(\pm 60^\circ)$.

If $z^{20} = \operatorname{cis} 60^\circ$, then $z^{20} - 1 = \operatorname{cis} 120^\circ$, so $z^8 = \operatorname{cis}(-120^\circ)$ and

$$z^4 = \frac{z^{20}}{(z^8)^2} = \operatorname{cis}(60^\circ + 240^\circ) = \operatorname{cis} 300^\circ,$$

which means $4\theta \equiv 300^\circ$, i.e. $\theta \equiv 75^\circ \pmod{90^\circ}$. Conversely every such θ works, since then $z^{20} = (z^4)^5 = \operatorname{cis} 60^\circ$ and $z^8 = (z^4)^2 = \operatorname{cis}(-120^\circ)$. The case $z^{20} = \operatorname{cis}(-60^\circ)$ similarly gives exactly $\theta \equiv 15^\circ \pmod{90^\circ}$.

So the $2n = 8$ angles in increasing order are $15, 75, 105, 165, 195, 255, 285, 345$, and $\theta_2 + \theta_4 + \theta_6 + \theta_8 = 75 + 165 + 255 + 345 = 840$.

15. Let $EFGH$, $EFDC$, and $EHBC$ be three adjacent square faces of a cube, for which $EC = 8$, and let A be the eighth vertex of the cube. Let I, J, and K be points on $\overline{EF}$, $\overline{EH}$, and $\overline{EC}$, respectively, so that $EI = EJ = EK = 2$. A solid S is obtained by drilling a tunnel through the cube. The sides of the tunnel are planes parallel to $\overline{AE}$, and containing the edges $\overline{IJ}$, $\overline{JK}$, and $\overline{KI}$. The surface area of S, including the walls of the tunnel, is $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. Find $m + n + p$.

Solution:

Place $A = (0,0,0)$ and $E = (8,8,8)$, so that $I = (6,8,8)$, $J = (8,6,8)$, $K = (8,8,6)$, and $\overline{AE}$ has direction $(1,1,1)$. The line through I in that direction leaves the cube at $L = (0,2,2)$; similarly J and K lead to $M = (2,0,2)$ and $N = (2,2,0)$. The tunnel wall through I and J is the plane $2z = x + y + 2$, which also contains L and M and crosses the z-axis at $O = (0,0,1)$; the other two walls behave symmetrically, crossing the y- and x-axes at $(0,1,0)$ and $(1,0,0)$.

Now add up the surface. Each of the three cube faces at E loses a right triangle with legs 2 (such as IEJ), leaving area $64 - 2 = 62$. Each of the three faces at A loses a quadrilateral of area 2 : on the face $z = 0$ its vertices are $(0,0,0)$, $(1,0,0)$, $(2,2,0)$, $(0,1,0)$. Each tunnel wall is a pentagon like $ILOMJ$: the rectangle $ILMJ$ with $IJ = 2\sqrt{2}$ and $IL = 6\sqrt{3}$ has area $12\sqrt{6}$, and the isosceles triangle LOM with base $LM = 2\sqrt{2}$ and height $\sqrt{3}$ adds $\sqrt{6}$, for $13\sqrt{6}$ per wall.

The total surface area is $6 \cdot 62 + 3 \cdot 13\sqrt{6} = 372 + 39\sqrt{6}$, so $m + n + p = 372 + 39 + 6 = 417$.

Problems: https://live.poshenloh.com/past-contests/aime/2001II