2001 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Find the sum of all positive two-digit integers that are divisible by each of their digits.

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Solution:

Let the number be $10a + b$ with tens digit a and units digit b. Since $a \mid 10a + b$, we need $a \mid b$, so $b = ka$ for some positive integer k. Since $b \mid 10a + b$, we need $b \mid 10a$, that is $ka \mid 10a$, so $k \mid 10$. Because $b = ka \leq 9$, only $k = 1, 2,$ and 5 are possible.

For $k = 1$ the numbers are $11, 22, \ldots, 99$, with sum $11 \cdot 45 = 495$. For $k = 2$ they are $12, 24, 36, 48$, with sum 120. For $k = 5$ the only one is 15.

The total is $495 + 120 + 15 = 630$.

2. A finite set $\mathcal{S}$ of distinct real numbers has the following properties: the mean of $\mathcal{S} \cup \{1\}$ is 13 less than the mean of $\mathcal{S}$, and the mean of $\mathcal{S} \cup \{2001\}$ is 27 more than the mean of $\mathcal{S}$. Find the mean of $\mathcal{S}$.

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Solution:

Let $\mathcal{S}$ have n elements with mean x, so the elements sum to nx. The two conditions say

$$\frac{nx+1}{n+1} = x - 13 \qquad \text{and} \qquad \frac{nx+2001}{n+1} = x + 27.$$

Clearing denominators, $nx + 1 = (n+1)x - 13(n+1)$ gives $x - 13(n+1) = 1$, and $nx + 2001 = (n+1)x + 27(n+1)$ gives $x + 27(n+1) = 2001$. Subtracting the first equation from the second yields $40(n+1) = 2000$, so $n + 1 = 50$.

Then $x = 1 + 13 \cdot 50 = 651$.

3. Find the sum of the roots, real and non-real, of the equation

$$x^{2001} + \left(\tfrac{1}{2} - x\right)^{2001} = 0,$$

given that there are no multiple roots.

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Solution:

Expand $\left(\frac{1}{2} - x\right)^{2001}$ by the binomial theorem. Its leading term $(-x)^{2001} = -x^{2001}$ cancels the x^{2001} in the equation, so what remains is a polynomial of degree 2000 :

$$2001 \cdot \frac{1}{2}\, x^{2000} - \binom{2001}{2} \frac{1}{4}\, x^{1999} + \cdots = 0.$$

By Vieta's formulas, the sum of the 2000 roots is

$$\frac{\binom{2001}{2}/4}{2001/2} = \frac{2001 \cdot 2000/8}{2001/2} = \frac{2000}{4} = 500.$$

4. In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where $a, b,$ and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

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Solution:

Since $\angle A = 60^\circ$ and $\angle B = 45^\circ$, we have $\angle C = 75^\circ$. In triangle ATC, angle $TAC = 30^\circ$ (half of angle A), so $\angle ATC = 180^\circ - 30^\circ - 75^\circ = 75^\circ$. Thus triangle ACT is isosceles with $AC = AT = 24$.

Drop the altitude CH to $\overline{AB}$. Triangle ACH is 30-60-90, so $AH = 12$ and $CH = 12\sqrt{3}$. Triangle BCH is 45-45-90, so $BH = CH = 12\sqrt{3}$.

The area is $\frac{1}{2} \cdot CH \cdot AB = \frac{1}{2} \cdot 12\sqrt{3}\,(12 + 12\sqrt{3}) = 216 + 72\sqrt{3}$. Then $a + b + c = 216 + 72 + 3 = 291$.

5. An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is $(0, 1)$, one altitude is contained in the y-axis, and the length of each side is $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Since one altitude lies along the y-axis, the other two vertices are symmetric: (x, y) and $(-x, y)$ with $x > 0$. The side from $(0, 1)$ to (x, y) makes a 120° angle with the positive x-axis, so it lies on the line $y = -\sqrt{3}\,x + 1$.

Substituting into $x^2 + 4y^2 = 4$ gives $x^2 + 4(1 - \sqrt{3}x)^2 = 4$, which simplifies to $13x^2 - 8\sqrt{3}\,x = 0$, so $x = \frac{8\sqrt{3}}{13}$.

The side length is $2x = \frac{16\sqrt{3}}{13}$, whose square is $\frac{768}{169}$. Since $\gcd(768, 169) = 1$, the answer is $768 + 169 = 937$.

6. A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

The rolls must form a non-decreasing sequence. Every multiset of four values from $\{1,\ldots,6\}$ can be arranged in non-decreasing order in exactly one way, so the number of successful outcomes equals the number of such multisets. By stars and bars (4 stars and 5 dividers), that count is $\binom{9}{4} = 126$.

The probability is $\frac{126}{6^4} = \frac{126}{1296} = \frac{7}{72}$, so $m+n = 7+72 = 79$.

7. Triangle ABC has $AB = 21$, $AC = 22$, and $BC = 20$. Points D and E are located on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$ and contains the center of the inscribed circle of triangle ABC. Then $DE = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

Since $\overline{DE} \parallel \overline{BC}$, triangles ADE and ABC are similar, and the ratio equals the ratio of their heights from A. The line DE passes through the incenter, which sits at height r (the inradius) above BC, so the ratio is $\frac{h-r}{h} = 1 - \frac{r}{h}$, where h is the height from A to $\overline{BC}$.

If K is the area and $s = \frac{21+22+20}{2} = \frac{63}{2}$ the semiperimeter, then $r = \frac{K}{s}$ and $h = \frac{2K}{20}$, so

$$\frac{r}{h} = \frac{20}{2s} = \frac{20}{63}.$$

Therefore $DE = 20\left(1 - \frac{20}{63}\right) = 20 \cdot \frac{43}{63} = \frac{860}{63}$, which is in lowest terms, and $m + n = 860 + 63 = 923$.

8. Call a positive integer N a *7–10 double* if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7–10 double because its base-7 representation is 102. What is the largest 7–10 double?

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Solution:

Suppose N has base-7 digits $d_k \ldots d_1 d_0$. The condition is $\sum d_i\,10^i = 2\sum d_i\,7^i$, that is $\sum d_i\,(10^i - 2\cdot 7^i) = 0$. The coefficients $10^i - 2\cdot 7^i$ for $i = 0, 1, 2, 3$ are $-1, -4, 2,$ 314. If there were a digit $d_3 \geq 1$, the positive contribution would be at least 314, but the negative terms total at most $4\cdot 6 + 6 = 30$. So N has at most three base-7 digits.

For three digits the condition reads $2d_2 = 4d_1 + d_0$. To maximize $N = 49d_2 + 7d_1 + d_0$, take $d_2 = 6$, so $4d_1 + d_0 = 12$; the largest value of $7d_1 + d_0$ comes from $d_1 = 3$, $d_0 = 0$.

Thus $N = 49\cdot 6 + 7\cdot 3 = 315$, whose base-$7$ representation is $630 = 2\cdot 315$.

9. In triangle ABC, $AB = 13$, $BC = 15$, and $CA = 17$. Point D is on $\overline{AB}$, E is on $\overline{BC}$, and F is on $\overline{CA}$. Let $AD = p \cdot AB$, $BE = q \cdot BC$, and $CF = r \cdot CA$, where $p, q,$ and r are positive and satisfy $p + q + r = \frac{2}{3}$ and $p^2 + q^2 + r^2 = \frac{2}{5}$. The ratio of the area of triangle DEF to the area of triangle ABC can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Each corner triangle's area is a product of side fractions: $[ADF] = p(1-r)[ABC]$, $[BED] = q(1-p)[ABC]$, and $[CFE] = r(1-q)[ABC]$, using the formula $\frac{1}{2}xy\sin\theta$ on the shared angles. Subtracting,

$$\frac{[DEF]}{[ABC]} = 1 - p(1-r) - q(1-p) - r(1-q) = 1 - (p+q+r) + (pq + qr + rp).$$

From the given values, $pq + qr + rp = \frac{(2/3)^2 - 2/5}{2} = \frac{4/9 - 2/5}{2} = \frac{1}{45}$.

Therefore the ratio is $1 - \frac{2}{3} + \frac{1}{45} = \frac{16}{45}$, and $m + n = 16 + 45 = 61$.

10. Let $\mathcal{S}$ be the set of points whose coordinates $x, y,$ and z are integers that satisfy $0 \leq x \leq 2, 0 \leq y \leq 3,$ and $0 \leq z \leq 4.$ Two distinct points are randomly chosen from $\mathcal{S}.$ The probability that the midpoint of the segment they determine also belongs to $\mathcal{S}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n.$

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Solution:

The midpoint is a lattice point exactly when the two chosen points agree in parity in each coordinate. Count ordered pairs (allowing equality) coordinate by coordinate. For $x \in \{0, 1, 2\}$ there are 2 even and 1 odd values, giving $2^2 + 1^2 = 5$ same-parity ordered pairs. For $y \in \{0, \ldots, 3\} : 2^2 + 2^2 = 8.$ For $z \in \{0, \ldots, 4\} : 3^2 + 2^2 = 13.$

That gives $5 \cdot 8 \cdot 13 = 520$ ordered pairs, including the 60 pairs where the two points are equal, so $520 - 60 = 460$ ordered pairs of distinct points, or 230 unordered pairs. The total number of unordered pairs is $\binom{60}{2} = 1770.$

The probability is $\frac{230}{1770} = \frac{23}{177}$, and since $177 = 3 \cdot 59,$ this is in lowest terms. Thus $m + n = 23 + 177 = 200.$

11. In a rectangular array of points, with 5 rows and N columns, the points are numbered consecutively from left to right beginning with the top row. Thus the top row is numbered 1 through N, the second row is numbered $N+1$ through $2N$, and so forth. Five points, P_1, P_2, P_3, P_4, and P_5, are selected so that each P_i is in row i. Let x_i be the number associated with P_i. Now renumber the array consecutively from top to bottom, beginning with the first column. Let y_i be the number associated with P_i after renumbering.

It is found that $x_1 = y_2, x_2 = y_1, x_3 = y_4, x_4 = y_5$, and $x_5 = y_3$. Find the smallest possible value of N.

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Solution:

Let P_i sit in column c_i. Then $x_i = (i-1)N + c_i$ and $y_i = 5(c_i - 1) + i$. The five conditions become

$$c_1 = 5c_2 - 3, \quad N + c_2 = 5c_1 - 4, \quad 2N + c_3 = 5c_4 - 1,$$

$$3N + c_4 = 5c_5, \quad 4N + c_5 = 5c_3 - 2.$$

Substituting $c_1 = 5c_2 - 3$ into the second equation gives $N = 24c_2 - 19$. Eliminating c_3 and c_4 from the last three equations yields $124c_5 = 89N + 7$. Substituting $N = 24c_2 - 19$ and reducing, $31 \mid 534c_2 - 421$, i.e. $7c_2 \equiv 18 \pmod{31}$, whose smallest positive solution is $c_2 = 7$.

Then $N = 24 \cdot 7 - 19 = 149$, and back-substituting gives valid columns $(c_1, \ldots, c_5) = (32, 7, 107, 45, 86)$, all at most 149. So the smallest possible N is 149.

12. A sphere is inscribed in the tetrahedron whose vertices are $A = (6, 0, 0)$, $B = (0, 4, 0)$, $C = (0, 0, 2)$, and $D = (0, 0, 0)$. The radius of the sphere is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Connecting the incenter to the four vertices splits the tetrahedron into four pyramids of height r over the faces, so $V = \frac{1}{3}rS$, i.e. $r = \frac{3V}{S}$, where S is the total surface area.

Here $V = \frac{1}{6} \cdot 6 \cdot 4 \cdot 2 = 8$. The three faces on the coordinate planes are right triangles with areas $\frac{1}{2} \cdot 6 \cdot 4 = 12$, $\frac{1}{2} \cdot 6 \cdot 2 = 6$, and $\frac{1}{2} \cdot 4 \cdot 2 = 4$. For face ABC, the cross product of $\overrightarrow{AB} = (-6, 4, 0)$ and $\overrightarrow{AC} = (-6, 0, 2)$ is $(8, 12, 24)$, with length $4\sqrt{2^2 + 3^2 + 6^2} = 28$, so that face has area 14.

Then $S = 12 + 6 + 4 + 14 = 36$ and $r = \frac{24}{36} = \frac{2}{3}$, giving $m + n = 2 + 3 = 5$.

13. In a certain circle, the chord of a d-degree arc is 22 centimeters long, and the chord of a $2d$-degree arc is 20 centimeters longer than the chord of a $3d$-degree arc, where $d < 120$. The length of the chord of a $3d$-degree arc is $-m+\sqrt{n}$ centimeters, where m and n are positive integers. Find $m+n$.

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Solution:

A chord subtending a θ-degree arc in a circle of radius R has length $2R\sin\frac{\theta}{2}$. Write $t=\frac{d}{2}$, so the three chords are $2R\sin t = 22$, $2R\sin 2t$, and $2R\sin 3t$. Using $\sin 2t = 2\sin t\cos t$ and $\sin 3t = \sin t\left(4\cos^2 t - 1\right)$, the chords of the $2d$- and $3d$-degree arcs are $22\cdot 2\cos t = 44\cos t$ and $22\left(4\cos^2 t-1\right)$.

The condition "the $2d$-chord is 20 longer than the $3d$-chord" becomes $44\cos t = 22\left(4\cos^2 t - 1\right)+20$, which simplifies to $44\cos^2 t - 22\cos t - 1 = 0$. From this, $4\cos^2 t = 2\cos t + \frac{1}{11}$, so the $3d$-chord equals $22\left(2\cos t + \frac{1}{11} - 1\right) = 44\cos t - 20$.

Solving the quadratic, $\cos t = \frac{22+\sqrt{484+176}}{88} = \frac{11+\sqrt{165}}{44}$ (the $+$ root since $d < 120$ means $t < 60^\circ$, so $\cos t > \frac{1}{2}$). Then the $3d$-chord is $44\cos t - 20 = 11+\sqrt{165}-20 = -9+\sqrt{165}$, giving $m+n = 9+165 = 174$.

14. A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

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Solution:

Write 1 for a house that gets mail and 0 for one that does not. Valid patterns are binary strings of length 19 with no two consecutive 1s and no three consecutive 0s. Let A_n, B_n, C_n count valid length-n strings ending in 1, in exactly one 0, and in exactly two 0s. A 1 may follow either kind of 0-ending, a single 0 may follow a 1, and a second 0 may follow a single 0 :

$$A_n = B_{n-1} + C_{n-1}, \qquad B_n = A_{n-1}, \qquad C_n = B_{n-1}.$$

Starting from $A_1 = B_1 = 1, C_1 = 0$ and iterating, the totals $A_n + B_n + C_n$ run $2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351.$

For $n = 19$ the count is $351.$

15. The numbers $1, 2, 3, 4, 5, 6, 7,$ and 8 are randomly written on the faces of a regular octahedron so that each face contains a different number. The probability that no two consecutive numbers, where 8 and 1 are considered to be consecutive, are written on faces that share an edge is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n.$

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Solution:

Pass to the dual cube: the octahedron's faces correspond to a cube's vertices, and two faces share an edge exactly when the corresponding cube vertices are adjacent. Following the numbers $1, 2, \ldots, 8$ and back to 1 traces a closed 8-step circuit through all the cube's vertices, and the requirement is that every step is a *diagonal* (an edge of one of the two inscribed tetrahedra, or one of the 4 long space diagonals). There are 16 such diagonals.

Each vertex lies on exactly one long diagonal, so the circuit cannot take two long diagonals in a row, and switching between the two tetrahedra is possible only via a long diagonal. Hence the circuit uses either 4 long diagonals alternating with tetrahedron edges, or 2 long diagonals separated by 3-edge paths in each tetrahedron. In the first case, choosing a pair of opposite edges in each tetrahedron ($3 \cdot 2$ ways) gives 6 octagons, each traceable as $8 \cdot 2$ permutations: $96.$ In the second case, a 3-edge path in one tetrahedron can be chosen in $4! = 24$ ways, and the return path through the other tetrahedron is then forced up to 2 choices, giving $8 \cdot 24 \cdot 2 = 384$ permutations.

So $96 + 384 = 480$ of the $8! = 40320$ labelings work, and the probability is $\frac{480}{40320} = \frac{1}{84}$. Thus $m + n = 1 + 84 = 85.$

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