2000 AIME II Solutions

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1. The number

$$\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Since $\frac{1}{\log_b a} = \log_a b$, the two terms equal $2 \log_{2000^6} 4 = \log_{2000^6} 16$ and $3 \log_{2000^6} 5 = \log_{2000^6} 125$. Their sum is

$$\log_{2000^6}(16 \cdot 125) = \log_{2000^6} 2000 = \frac{1}{6}.$$

Since $\gcd(1, 6) = 1$, the answer is $m + n = 1 + 6 = 7$.

2. A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

Solution:

Factor $(x-y)(x+y) = 2000^2 = 2^8 \cdot 5^6$. The factors $x-y$ and $x+y$ have the same parity, and their product is even, so both must be even. Writing $x - y = 2a$ and $x + y = 2b$ gives $ab = 2^6 \cdot 5^6 = 10^6$.

Each ordered pair of positive integers (a, b) with $ab = 10^6$ yields exactly one solution $x = a + b, y = b - a$ with $x > 0$, and 10^6 has $7 \cdot 7 = 49$ divisors, hence 49 such pairs. Replacing (a, b) by $(-a, -b)$ gives the 49 solutions with $x < 0$, and $x = 0$ is impossible since $-y^2 < 2000^2$.

In total there are $49 + 49 = 98$ lattice points.

3. A deck of forty cards consists of four 1's, four 2's, $\ldots$, and four 10's. A matching pair (two cards with the same number) is removed from the deck. Given that these cards are not returned to the deck, let m/n be the probability that two randomly selected cards also form a pair, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

After the matching pair is removed, 38 cards remain: nine numbers with four cards each and one number with only two cards. The number of ways to draw a pair is $9\binom{4}{2} + \binom{2}{2} = 54 + 1 = 55$, out of $\binom{38}{2} = 703$ equally likely draws.

Since $703 = 19 \cdot 37$ shares no factor with $55 = 5 \cdot 11$, the probability $\frac{55}{703}$ is in lowest terms, and $m + n = 55 + 703 = 758$.

4. What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?

Solution:

Write $N = 2^a m$ with m odd. The odd divisors of N are exactly the divisors of m, so $d(m) = 6$. Every even divisor is 2^k (for $1 \le k \le a$) times an odd divisor, so there are $a \cdot d(m) = 6a$ of them, and $6a = 12$ gives $a = 2$.

So $N = 4m$ where m is the smallest odd number with exactly 6 divisors. The shapes are p^5 (smallest $3^5 = 243$) and $p^2 q$ (smallest $3^2 \cdot 5 = 45$), so $m = 45$ and $N = 4 \cdot 45 = 180.$

5. Given eight distinguishable rings, let n be the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. The order of rings on each finger is significant, but it is not required that each finger have a ring. Find the leftmost three nonzero digits of n.

Solution:

Choose which five rings to use in $\binom{8}{5} = 56$ ways, and order them (reading down the first finger, then the second, and so on) in $5! = 120$ ways. It remains to split the ordered list into four possibly empty consecutive blocks, one per finger: the number of compositions of 5 into 4 nonnegative parts, which by stars and bars is $\binom{8}{3} = 56.$

Therefore $n = 56 \cdot 120 \cdot 56 = 376320,$ whose leftmost three nonzero digits are $376.$

6. One base of a trapezoid is 100 units longer than the other base. The segment that joins the midpoints of the legs divides the trapezoid into two regions whose areas are in the ratio $2:3$. Let x be the length of the segment joining the legs of the trapezoid that is parallel to the bases and that divides the trapezoid into two regions of equal area. Find the greatest integer that does not exceed $x^2/100$.

Solution:

Let the bases be b and $b+100$. The midsegment has length $b+50$ and splits the trapezoid into two trapezoids of equal height, whose areas are proportional to the sums of their parallel sides, $b+(b+50)$ and $(b+50)+(b+100)$. Setting $\frac{2b+50}{2b+150}=\frac{2}{3}$ gives $b=75$, so the bases are 75 and 175.

Extend the legs to meet at an apex, creating similar triangles: a segment parallel to the bases with length ℓ cuts off a triangle of area $c\ell^2$ for a fixed constant c. The segment of length x bisects the trapezoid's area exactly when $cx^2 - c\cdot 75^2 = c\cdot 175^2 - cx^2$, so

$$x^2 = \frac{75^2+175^2}{2} = 18125.$$

Then $x^2/100 = 181.25$, and the greatest integer not exceeding it is 181.

7. Given that

$$\frac{1}{2!17!}+\frac{1}{3!16!}+\frac{1}{4!15!}+\frac{1}{5!14!}+\frac{1}{6!13!}+\frac{1}{7!12!}+\frac{1}{8!11!}+\frac{1}{9!10!}=\frac{N}{1!18!},$$

find the greatest integer that is less than $\frac{N}{100}$.

Solution:

Multiply both sides by $19!$. Each term on the left becomes $\frac{19!}{k!\,(19-k)!}=\binom{19}{k}$ for $k=2,\ldots,9$, while the right side becomes $19N$.

Since $\binom{19}{k}=\binom{19}{19-k}$, the first half of the binomial row sums to $\sum_{k=0}^{9}\binom{19}{k}=\frac{2^{19}}{2}=2^{18}$, so

$$\sum_{k=2}^{9}\binom{19}{k}=2^{18}-\binom{19}{0}-\binom{19}{1}=262144-20=262124.$$

Hence $N=\frac{262124}{19}=13796$, so $\frac{N}{100}=137.96$, and the greatest integer less than this is 137.

8. In trapezoid $ABCD$, leg $\overline{BC}$ is perpendicular to bases $\overline{AB}$ and $\overline{CD}$, and diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. Given that $AB = \sqrt{11}$ and $AD = \sqrt{1001}$, find BC^2.

Solution:

Place $B = (0, 0)$, $A = (\sqrt{11}, 0)$, $C = (0, h)$, and $D = (d, h)$, so that $\overline{BC}$ is vertical and $BC^2 = h^2$. The diagonals give vectors $\overrightarrow{AC} = (-\sqrt{11}, h)$ and $\overrightarrow{BD} = (d, h)$, and perpendicularity means $-\sqrt{11}\,d + h^2 = 0$, so $d = \frac{h^2}{\sqrt{11}}$.

Then $AD^2 = (d - \sqrt{11})^2 + h^2 = 1001$. Setting $u = h^2$, this becomes $\frac{(u-11)^2}{11} + u = 1001$, that is,

$$u^2 - 11u - 10890 = 0.$$

The positive root is $u = \frac{11+\sqrt{121+43560}}{2} = \frac{11+209}{2} = 110$, so $BC^2 = 110$.

9. Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

Solution:

From $z + \frac{1}{z} = 2\cos 3^\circ$ we get $z^2 - (2\cos 3^\circ)z + 1 = 0$, so $z = \cos 3^\circ \pm i \sin 3^\circ$, a point on the unit circle. By de Moivre's theorem,

$$z^{2000} + \frac{1}{z^{2000}} = 2\cos(2000 \cdot 3^\circ) = 2\cos 6000^\circ.$$

Since $6000 = 16 \cdot 360 + 240$, this equals $2\cos 240^\circ = -1$. The least integer greater than -1 is 0.

10. A circle is inscribed in quadrilateral $ABCD$, tangent to $\overline{AB}$ at P and to $\overline{CD}$ at Q. Given that $AP = 19$, $PB = 26$, $CQ = 37$, and $QD = 23$, find the square of the radius of the circle.

Solution:

Let the incircle have center I and radius r. The tangent lengths from A, B, C, D are $19, 26, 37, 23$, and I lies on each angle bisector, so the half-angles $\alpha, \beta, \gamma, \delta$ at the four vertices satisfy $\tan\alpha = \frac{r}{19}, \tan\beta = \frac{r}{26}, \tan\gamma = \frac{r}{37}, \tan\delta = \frac{r}{23}$, with $\alpha + \beta + \gamma + \delta = 180^\circ$.

Then $\tan(\alpha+\gamma) = -\tan(\beta+\delta)$, and the tangent addition formula turns this into

$$\frac{\frac{r}{19}+\frac{r}{37}}{1-\frac{r^2}{19\cdot 37}} = -\frac{\frac{r}{26}+\frac{r}{23}}{1-\frac{r^2}{26\cdot 23}}, \qquad \text{i.e.} \qquad \frac{56r}{703-r^2} = \frac{49r}{r^2-598}.$$

Cross-multiplying gives $56r^2 - 56\cdot 598 = 49\cdot 703 - 49r^2$, so $105r^2 = 33488 + 34447 = 67935$ and $r^2 = 647$.

11. The coordinates of the vertices of isosceles trapezoid $ABCD$ are all integers, with $A = (20, 100)$ and $D = (21, 107)$. The trapezoid has no horizontal or vertical sides, and $\overline{AB}$ and $\overline{CD}$ are the only parallel sides. The sum of the absolute values of all possible slopes for $\overline{AB}$ is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Since all vertices are lattice points, $w = \overrightarrow{BC}$ is an integer vector with $|w| = |\overrightarrow{AD}| = \sqrt{50}$, so w is one of $(\pm 1, \pm 7), (\pm 7, \pm 1), (\pm 5, \pm 5)$. Write $\overrightarrow{AD} = (1, 7) = s\,\hat{u} + h\,\hat{v}$ where $\hat{u}$ points along $\overline{AB}$ and $\hat{v}$ is perpendicular. Because $\overline{AB} \parallel \overline{CD}$, the vector w has the same perpendicular component h, and the equal leg lengths force its $\hat{u}$-component to be $-s$ (the value $+s$ gives a parallelogram). Hence $(1, 7) - w = 2s\,\hat{u}$ is parallel to $\overline{AB}$.

Discard $w = (1, 7)$ (parallelogram) and $w = (-1, -7)$ (then $h = 0$, degenerate). The choices $w = (1, -7)$ and $w = (-1, 7)$ make $(1, 7) - w$ vertical or horizontal, which is forbidden. The remaining eight choices give $(1, 7) - w$ equal to $(-6, 6)$, $(-6, 8)$, $(8, 6)$, $(8, 8)$, $(-4, 2)$, $(-4, 12)$, $(6, 2)$, $(6, 12)$, with slopes $-1, -\frac{4}{3}, \frac{3}{4}, 1, -\frac{1}{2}, -3, \frac{1}{3}, 2$; each is realizable by placing B suitably far along $\hat{u}$.

The sum of the absolute values is $1 + \frac{4}{3} + \frac{3}{4} + 1 + \frac{1}{2} + 3 + \frac{1}{3} + 2 = \frac{119}{12}$, so $m + n = 119 + 12 = 131$.

12. The points A, B, and C lie on the surface of a sphere with center O and radius 20. It is given that $AB = 13, BC = 14, CA = 15$, and that the distance from O to triangle ABC is $\frac{m\sqrt{n}}{k}$, where m, n, and k are positive integers, m and k are relatively prime, and n is not divisible by the square of any prime. Find $m + n + k$.

Solution:

The foot of the perpendicular from O to the plane of ABC is equidistant from A, B, and C (the slant segments to the vertices all have length 20), so it is the circumcenter of triangle ABC.

By Heron's formula with $s = 21$, the area is $K = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, so the circumradius is $R = \frac{abc}{4K} = \frac{13 \cdot 14 \cdot 15}{336} = \frac{65}{8}$. The distance from O to the plane is

$$\sqrt{20^2 - \left(\tfrac{65}{8}\right)^2} = \sqrt{\frac{25600 - 4225}{64}} = \frac{\sqrt{21375}}{8} = \frac{15\sqrt{95}}{8}.$$

Here $\gcd(15, 8) = 1$ and $95 = 5 \cdot 19$ is squarefree, so $m + n + k = 15 + 95 + 8 = 118.$

13. The equation $2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$ has exactly two real roots, one of which is $\frac{m+\sqrt{n}}{r}$, where $m, n,$ and r are integers, m and r are relatively prime, and $r > 0$. Find $m + n + r$.

Solution:

Group the equation as

$$2(1000x^6 - 1) + x(100x^4 + 10x^2 + 1) = 0.$$

Since $1000x^6 - 1 = (10x^2)^3 - 1 = (10x^2 - 1)(100x^4 + 10x^2 + 1)$, the left side factors as

$$(100x^4 + 10x^2 + 1)\big(2(10x^2 - 1) + x\big) = (100x^4 + 10x^2 + 1)(20x^2 + x - 2).$$

The quartic factor is always positive, so the two real roots are the roots of $20x^2 + x - 2 = 0$, namely $x = \frac{-1 \pm \sqrt{161}}{40}$. The root of the form $\frac{m+\sqrt{n}}{r}$ is $\frac{-1+\sqrt{161}}{40}$, with $m = -1$, $n = 161$, $r = 40$, and $\gcd(-1, 40) = 1$. Thus $m + n + r = -1 + 161 + 40 = 200.$

14. Every positive integer k has a unique factorial base expansion $(f_1, f_2, f_3, \ldots, f_m)$, meaning that $k = 1! \cdot f_1 + 2! \cdot f_2 + 3! \cdot f_3 + \cdots + m! \cdot f_m$, where each f_i is an integer, $0 \le f_i \le i$, and $0 < f_m$. Given that $(f_1, f_2, f_3, \ldots, f_j)$ is the factorial base expansion of

$$16! - 32! + 48! - 64! + \cdots + 1968! - 1984! + 2000!,$$

find the value of $f_1 - f_2 + f_3 - f_4 + \cdots + (-1)^{j+1} f_j$.

Solution:

Since $(i+1)! - i! = i \cdot i!$, telescoping gives $a! - b! = \sum_{i=b}^{a-1} i \cdot i!$ for $a > b$. Group the given number as

$$16! + (48! - 32!) + (80! - 64!) + \cdots + (2000! - 1984!),$$

with 62 parenthesized groups $(32j + 16)! - (32j)!$ for $j = 1, \ldots, 62$.

The group for j contributes factorial-base digits $f_i = i$ for $32j \le i \le 32j + 15$, and the lone $16!$ contributes $f_{16} = 1$; all other digits are 0. Every digit satisfies $0 \le f_i \le i$, so by uniqueness this is the factorial base expansion.

In the alternating sum, $f_{16} = 1$ sits at an even index and contributes -1. Each group's range starts at an even index and has length 16, so it splits into 8 consecutive pairs, each contributing $-i + (i+1) = 1$, for $+8$ per group. The total is $62 \cdot 8 - 1 = 495$.

15. Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

Solution:

Since $\sin 1^\circ = \sin\big((k+1)^\circ - k^\circ\big) = \sin(k+1)^\circ \cos k^\circ - \cos(k+1)^\circ \sin k^\circ$, dividing by $\sin k^\circ \sin(k+1)^\circ$ gives

$$\frac{1}{\sin k^\circ \sin(k+1)^\circ} = \frac{\cot k^\circ - \cot(k+1)^\circ}{\sin 1^\circ}.$$

So the sum times $\sin 1^\circ$ equals $\cot 45^\circ - \cot 46^\circ + \cot 47^\circ - \cot 48^\circ + \cdots + \cot 133^\circ - \cot 134^\circ$, with $+$ signs on odd arguments and $-$ signs on even arguments.

Because $\cot(180^\circ - x) = -\cot x$ and supplementary arguments here have the same parity, the terms cancel in supplementary pairs: $+\cot 133^\circ$ cancels $+\cot 47^\circ$, $-\cot 134^\circ$ cancels $-\cot 46^\circ$, and so on for every pair of arguments summing to 180°. The only survivors are $\cot 45^\circ = 1$ (its partner 135° is out of range) and $-\cot 90^\circ = 0$.

Hence the sum equals $\frac{\cot 45^\circ}{\sin 1^\circ} = \frac{1}{\sin 1^\circ}$, so the least such n is 1.

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