2000 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Find the least positive integer n such that no matter how 10^n is expressed as the product of any two positive integers, at least one of these two integers contains the digit 0.

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Solution:

Every factorization is $10^n = (2^x 5^y)(2^{n-x} 5^{n-y})$. If a factor is divisible by both 2 and 5, it is a multiple of 10 and ends in the digit 0. So the only possible zero-free factorization is $10^n = 2^n \cdot 5^n$, and we need the least n for which 2^n or 5^n contains a digit 0.

The powers $2^1, \ldots, 2^8$ are $2, 4, 8, 16, 32, 64, 128, 256$ — no zeros. The powers $5^1, \ldots, 5^7$ are $5, 25, 125, 625, 3125, 15625, 78125$ — no zeros — but $5^8 = 390625$ contains a 0.

Hence every factorization of 10^8 contains a digit 0, while $10^7 = 2^7 \cdot 5^7 = 128 \cdot 78125$ does not, so the answer is 8.

2. Let u and v be integers satisfying $0 < v < u$. Let $A = (u, v)$, let B be the reflection of A across the line $y = x$, let C be the reflection of B across the y-axis, let D be the reflection of C across the x-axis, and let E be the reflection of D across the y-axis. The area of pentagon $ABCDE$ is 451. Find $u + v$.

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Solution:

Carrying out the reflections, $B = (v, u)$, $C = (-v, u)$, $D = (-v, -u)$, and $E = (v, -u)$. The points B, C, D, E form a rectangle of width $2v$ and height $2u$, with area $4uv$, and $A = (u, v)$ sticks out to its right. Triangle ABE has vertical base BE of length $2u$ and horizontal height $u - v$, so its area is $u(u - v)$.

The pentagon's area is therefore

$$4uv + u(u - v) = u^2 + 3uv = u(u + 3v) = 451 = 11 \cdot 41.$$

Since $0 < v < u$, we have $u < u + 3v < 4u$, which rules out the factorization $1 \cdot 451$. So $u = 11$ and $u + 3v = 41$, giving $v = 10$, which indeed satisfies $v < u$.

Thus $u + v = 11 + 10 = 21$.

3. In the expansion of $(ax + b)^{2000}$, where a and b are relatively prime positive integers, the coefficients of x^2 and x^3 are equal. Find $a + b$.

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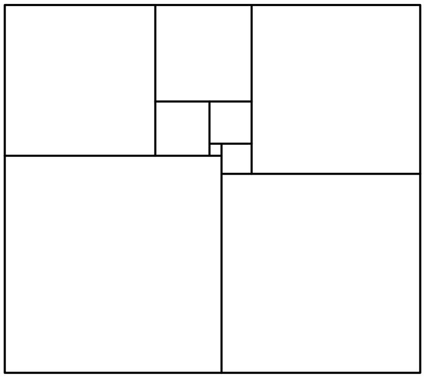
Solution:

By the binomial theorem, the coefficients of x^2 and x^3 are $\binom{2000}{2}a^2b^{1998}$ and $\binom{2000}{3}a^3b^{1997}$. Setting them equal and cancelling a^2b^{1997} gives

$$\binom{2000}{2}b = \binom{2000}{3}a, \qquad \text{so} \qquad b = \frac{1998}{3}a = 666a.$$

Since $\gcd(a, b) = 1$, we must have $a = 1$ and $b = 666$, so $a + b = 667$.

4. The diagram shows a rectangle that has been dissected into nine non-overlapping squares. Given that the width and the height of the rectangle are relatively prime positive integers, find the perimeter of the rectangle.



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Solution:

Let the tiniest square (in the middle) have side x and the small square just below and to its right have side y. Chasing edge lengths through the figure, the remaining squares have sides $x+y$, then $(x+y)+x=2x+y$, then $(x+y)+(2x+y)=3x+2y$, then $(2x+y)+(3x+2y)=5x+3y$ (the top-left square). The tall square on the right spans the previous three along its left edge minus overlaps, giving side $4x+4y$; the bottom-right square has side $(4x+4y)+y=4x+5y$; and the bottom-left square has side $x+(2x+y)+(5x+3y)=8x+4y$.

Measuring the rectangle's height along its left and right sides,

$$(8x+4y)+(5x+3y)=(4x+5y)+(4x+4y),$$

which simplifies to $5x=2y$. Taking the smallest positive integers, $x=2$ and $y=5$, the nine squares have sides $2, 5, 7, 9, 16, 25, 28, 33, 36$, and the rectangle is $(36+33)\times(36+25)=69\times 61$. These dimensions are relatively prime (any common scaling would break that), and the areas check: $69\cdot 61=4209$ equals the sum of the nine squares' areas.

The perimeter is $2(69+61)=260$.

5. Each of two boxes contains both black and white marbles, and the total number of marbles in the two boxes is 25. One marble is taken out of each box randomly. The probability that both marbles are black is $\frac{27}{50}$, and the probability that both marbles are white is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

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Solution:

Say the boxes hold a and b marbles with $a + b = 25$, containing p and q black marbles. Then $\frac{pq}{ab} = \frac{27}{50}$, so $50pq = 27ab$, and since $\gcd(27, 50) = 1$, we need $50 \mid ab$. Checking $a(25 - a)$ for $a = 1, \ldots, 12$, only $\{a, b\} = \{20, 5\}$ and $\{10, 15\}$ give a multiple of 50.

For sizes 20 and $5 : pq = \frac{27 \cdot 100}{50} = 54$, and since each box also holds a white marble, $p \leq 19$ and $q \leq 4$, forcing $p = 18, q = 3$. The white counts are 2 and 2, so the white-white probability is $\frac{2}{20} \cdot \frac{2}{5} = \frac{1}{25}$. For sizes 10 and $15 : pq = \frac{27 \cdot 150}{50} = 81$, and $p \leq 9, q \leq 14$ force $p = q = 9$. The white counts are 1 and 6, giving $\frac{1}{10} \cdot \frac{6}{15} = \frac{1}{25}$ again.

Either way the probability is $\frac{1}{25}$, so $m + n = 1 + 25 = 26$.

6. For how many ordered pairs (x, y) of integers is it true that $0 < x < y < 10^6$ and that the arithmetic mean of x and y is exactly 2 more than the geometric mean of x and y?

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Solution:

The condition is $\frac{x+y}{2} = \sqrt{xy} + 2$, that is, $x + y - 2\sqrt{xy} = 4$, so $(\sqrt{y} - \sqrt{x})^2 = 4$ and (as $y > x$) $\sqrt{y} - \sqrt{x} = 2$. Note $\sqrt{xy} = \frac{x+y-4}{2}$ is rational, hence $\sqrt{y} + \sqrt{x} = \frac{y-x}{\sqrt{y}-\sqrt{x}} = \frac{y-x}{2}$ is rational too, so $\sqrt{x}$ and $\sqrt{y}$ are rational — and a rational square root of an integer is an integer.

Therefore $x = a^2$ and $y = (a + 2)^2$ for a positive integer a. The constraint $y < 10^6$ means $a + 2 \leq 999$, so a ranges over $1, 2, \ldots, 997$, and each value gives a valid pair.

Hence there are 997 ordered pairs.

7. Suppose that $x, y,$ and z are three positive numbers that satisfy the equations $xyz = 1$, $x + \frac{1}{z} = 5$, and $y + \frac{1}{x} = 29$. Then $z + \frac{1}{y} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Let $t = z + \frac{1}{y}$. Expanding the product of all three expressions,

$$\left(x+\frac{1}{z}\right)\left(y+\frac{1}{x}\right)\left(z+\frac{1}{y}\right) = xyz + \frac{1}{xyz} + \left(x+\frac{1}{z}\right) + \left(y+\frac{1}{x}\right) + \left(z+\frac{1}{y}\right).$$

Since $xyz = 1$, the left side is $5 \cdot 29 \cdot t = 145t$ and the right side is $2 + 5 + 29 + t = 36 + t$.

So $145t = 36 + t$, giving $t = \frac{36}{144} = \frac{1}{4}$. Thus $m + n = 1 + 4 = 5$.

8. A container in the shape of a right circular cone is 12 inches tall and its base has a 5-inch radius. The liquid that is sealed inside is 9 inches deep when the cone is held with its point down and its base horizontal. When the cone is held with its point up and its base horizontal, the liquid is $m - n\sqrt[3]{p}$ inches deep, where $m, n,$ and p are positive integers and p is not divisible by the cube of any prime number. Find $m + n + p$.

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Solution:

Held point down, the liquid forms a cone similar to the container with ratio $\frac{9}{12} = \frac{3}{4}$, so its volume is $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$ of the container's volume.

Held point up, the empty space is a similar cone at the apex with $1 - \frac{27}{64} = \frac{37}{64}$ of the volume, so its height is

$$12\sqrt[3]{\frac{37}{64}} = \frac{12\sqrt[3]{37}}{4} = 3\sqrt[3]{37}$$

inches. The liquid is therefore $12 - 3\sqrt[3]{37}$ inches deep.

Since 37 is cube-free, $m + n + p = 12 + 3 + 37 = 52.$

9. The system of equations

$$\log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) = 4$$

$$\log_{10}(2yz) - (\log_{10} y)(\log_{10} z) = 1$$

$$\log_{10}(zx) - (\log_{10} z)(\log_{10} x) = 0$$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.

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Solution:

Let $a = \log_{10} x, b = \log_{10} y, c = \log_{10} z$. Using $\log_{10} 2000 = 4 - \log_{10} 5$, the first equation becomes $a + b - ab = \log_{10} 5$, which factors as $(1-a)(1-b) = 1 - \log_{10} 5 = \log_{10} 2$. Similarly the second equation gives $(1-b)(1-c) = 1 - (1 - \log_{10} 2) = \log_{10} 2$, and the third gives $(1-c)(1-a) = 1$.

Dividing the first two (note $1 - b \neq 0$) yields $1 - a = 1 - c$, and then the third equation gives $(1-a)^2 = 1$, so $1 - a = \pm 1$. If $1 - a = 1$, then $1 - b = \log_{10} 2$, so $b = 1 - \log_{10} 2 = \log_{10} 5$ and $y = 5$ (indeed $(x, y, z) = (1, 5, 1)$ works). If $1 - a = -1$, then $1 - b = -\log_{10} 2$, so $b = \log_{10} 20$ and $y = 20$ (from $(x, y, z) = (100, 20, 100)$).

Therefore $y_1 + y_2 = 5 + 20 = 25$.

10. A sequence of numbers $x_1, x_2, x_3, \ldots, x_{100}$ has the property that, for every integer k between 1 and $100,$ inclusive, the number x_k is k less than the sum of the other 99 numbers. Given that $x_{50} = \frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n.$

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Solution:

Let $S = x_1 + x_2 + \cdots + x_{100}$. The condition says $x_k = (S - x_k) - k$, so $x_k = \frac{S-k}{2}$ for every k. Summing over $k = 1, \ldots, 100$,

$$S = \frac{100S - (1 + 2 + \cdots + 100)}{2} = \frac{100S - 5050}{2},$$

so $98S = 5050$ and $S = \frac{2525}{49}$.

Then

$$x_{50} = \frac{S - 50}{2} = \frac{2525 - 2450}{98} = \frac{75}{98},$$

which is in lowest terms, so $m + n = 75 + 98 = 173.$

11. Let S be the sum of all numbers of the form $\frac{a}{b}$, where a and b are relatively prime positive divisors of 1000. What is the greatest integer that does not exceed $\frac{S}{10}$?

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Solution:

Write $a = 2^i 5^j$ and $b = 2^k 5^l$ with exponents between 0 and 3. Coprimality means $\min(i, k) = 0$ and $\min(j, l) = 0$, and these two constraints are independent. So as (a, b) runs over all coprime pairs, the factor 2^{i-k} independently takes each value in $\{2^{-3}, \dots, 2^3\}$ exactly once, and similarly for 5^{j-l}. Hence

$$S = \left(1 + 2 + 4 + 8 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\right)\left(1 + 5 + 25 + 125 + \tfrac{1}{5} + \tfrac{1}{25} + \tfrac{1}{125}\right) = \frac{127}{8} \cdot \frac{19531}{125}.$$

This equals $\frac{2480437}{1000} = 2480.437$, so $\frac{S}{10} = 248.0437$, and the greatest integer not exceeding it is 248.

12. Given a function f for which

$$f(x) = f(398 - x) = f(2158 - x) = f(3214 - x)$$

holds for all real x, what is the largest number of different values that can appear in the list $f(0), f(1), f(2), \ldots, f(999)$?

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Solution:

Since $f(398 - x) = f(2158 - x)$ for all x, substituting $t = 398 - x$ gives $f(t) = f(t + 1760)$; likewise $f(2158 - x) = f(3214 - x)$ gives period 1056. Combining, f has period $\gcd(1760, 1056) = 352$. Reducing 398 mod 352, the symmetry $f(x) = f(398 - x)$ becomes $f(x) = f(46 - x)$.

So f is determined by residues mod 352, with residues r and $46 - r$ forced to share a value. This pairing has exactly two fixed points, from $2r \equiv 46 \pmod{352} : r = 23$ and $r = 199$. Hence there are at most $\frac{352-2}{2} + 2 = 177$ classes, and since $0, 1, \ldots, 999$ covers every residue mod 352, the list contains at most 177 different values.

This is achievable: $f(x) = \cos \frac{2\pi(x-23)}{352}$ satisfies all three given symmetries (each of 398, $2158, 3214$ is $\equiv 46$ mod 352), and two integers get equal values only when their residues are paired. So the answer is 177.

13. In the middle of a vast prairie, a firetruck is stationed at the intersection of two perpendicular straight highways. The truck travels at 50 miles per hour along the highways and at 14 miles per hour across the prairie. Consider the set of points that can be reached by the firetruck within six minutes. The area of this region is $\frac{m}{n}$ square miles, where m and n are relatively prime positive integers. Find $m+n$.

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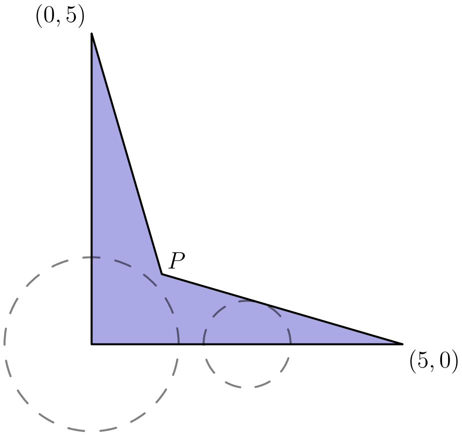
Solution:

In six minutes the truck can drive 5 miles on a highway or 1.4 miles across the prairie, and an optimal route is a highway stretch followed by a straight prairie segment. Work in the first quadrant with the highways as axes. Driving to $(d,0)$ takes $\frac{d}{50}$ hours, leaving a prairie range of $14\left(\frac{1}{10}-\frac{d}{50}\right)=1.4\left(1-\frac{d}{5}\right)$ miles. As d runs from 0 to 5, these disks shrink linearly to a point, so their union is the "cone": the convex hull of the disk of radius 1.4 about the origin and the point $(5,0)$, bounded by the tangent line from $(5,0)$. The tangent length is $\sqrt{5^2-1.4^2}=4.8$, so the ratios are 7-24-25 and the tangent line is $7x+24y=35$. The y-axis gives the mirror-image region bounded by $24x+7y=35$.



The two tangent lines meet at $P=\left(\frac{35}{31},\frac{35}{31}\right)$, which lies at distance $\frac{35\sqrt{2}}{31}>1.4$ from the origin — outside the circle — so in the first quadrant the reachable set is exactly the (non-convex) quadrilateral with vertices $(0,0)$, $(5,0)$, P, $(0,5)$. Splitting it along the diagonal from the origin to P gives two triangles, each with area $\frac{1}{2}\cdot 5\cdot\frac{35}{31}$, for a quadrant area of $\frac{175}{31}$.

The full region is four copies, with area $\frac{700}{31}$ square miles. Since $\gcd(700,31)=1$, the answer is $700+31=731$.

14. In triangle ABC, it is given that angles B and C are congruent. Points P and Q lie on $\overline{AC}$ and $\overline{AB}$, respectively, so that $AP = PQ = QB = BC$. Angle ACB is r times as large as angle APQ, where r is a positive real number. Find the greatest integer that does not exceed $1000r$.

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Solution:

Let $\angle A = \alpha$, and scale so $AP = PQ = QB = BC = 1$. In triangle APQ, the equal sides $AP = PQ$ give $\angle AQP = \angle A = \alpha$, so $\angle APQ = 180^\circ - 2\alpha$ and, by the law of sines, $AQ = \frac{\sin 2\alpha}{\sin \alpha} = 2\cos\alpha$. In triangle ABC, $\angle B = \angle C = 90^\circ - \frac{\alpha}{2}$, so

$$AB = \frac{BC \sin C}{\sin A} = \frac{\cos(\alpha/2)}{\sin \alpha} = \frac{1}{2\sin(\alpha/2)}.$$

Since $AQ + QB = AB$,

$$2\cos\alpha + 1 = \frac{1}{2\sin(\alpha/2)} \quad\implies\quad 4\sin\tfrac{\alpha}{2}\cos\alpha + 2\sin\tfrac{\alpha}{2} = 1.$$

By the product-to-sum identity, $4\sin\frac{\alpha}{2}\cos\alpha = 2\sin\frac{3\alpha}{2} - 2\sin\frac{\alpha}{2}$, so the equation collapses to $\sin\frac{3\alpha}{2} = \frac{1}{2}$. Then $\alpha = 20^\circ$ or $\alpha = 100^\circ$, but the latter makes $AQ = 2\cos\alpha$ negative, so $\alpha = 20^\circ$.

Now $\angle ACB = 80^\circ$ and $\angle APQ = 140^\circ$, so $r = \frac{80}{140} = \frac{4}{7}$, and $\lfloor 1000r \rfloor = \left\lfloor \frac{4000}{7} \right\rfloor = 571.$

15. A stack of 2000 cards is labelled with the integers from 1 to $2000,$ with different integers on different cards. The cards in the stack are not in numerical order. The top card is removed from the stack and placed on the table, and the next card is moved to the bottom of the stack. The new top card is removed from the stack and placed on the table, to the right of the card already there, and the next card in the stack is moved to the bottom of the stack. The process — placing the top card to the right of the cards already on the table and moving the next card in the stack to the bottom of the stack — is repeated until all cards are on the table. It is found that, reading from left to right, the labels on the cards are now in ascending order: $1, 2, 3, \ldots, 1999, 2000.$ In the original stack of cards, how many cards were above the card labelled $1999?$

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Solution:

Number the original positions 1 (top) through 2000 (bottom) and put them in a queue. Each step removes the front position (which receives the next label $1, 2, 3, \ldots$) and sends the new front to the back. So the card labelled 1999 is the next-to-last card removed, and we must find which original position survives that long.

The first pass removes the odd positions $1, 3, \ldots, 1999$ (labels 1 through 1000) and, since it ends by sending 2000 to the back, the next pass again starts by removing the front of the queue $2, 4, \ldots, 2000.$ Successive passes therefore remove $2, 6, \ldots, 1998$ (the positions $\equiv 2 \mod 4$), then $4, 12, \ldots, 1996,$ then $8, 24, \ldots, 1992,$ then $16, 48, \ldots, 1968, 2000$ (the 63 positions $\equiv 16 \mod 32$). That last pass ran through an odd number (125) of cards, so the alternation shifts: the 62 surviving multiples of 32 now sit in the queue as $64, 96, \ldots, 1984, 32.$

Continuing the same removal pattern from that queue, the next rounds remove $64, 128, \ldots, 1984;$ then $96, 224, \ldots, 1888, 32;$ then $288, 544, \ldots, 1824;$ then $160, 672, 1184, 1696;$ then $416, 1440;$ and the final two cards removed are 928 and $1952.$ So label 1999 goes to the card at original position $928,$ which had 927 cards above it.

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