1999 AIME Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/aime/1999/solutions

1. Find the smallest prime that is the fifth term of an increasing arithmetic sequence, all four preceding terms also being prime.

Solution:

Let the terms be $p, p+d, \ldots, p+4d$. If d were odd, consecutive terms would have opposite parity, so some term other than the first would be even and greater than 2 — impossible. If d were not a multiple of 3, then $p, p+d, p+2d$ would cover all residues mod 3, so some term would be divisible by 3; that term would have to be 3 itself, forcing $p = 3$, but then $p + 3d = 3(1+d)$ is composite. Hence $6 \mid d$.

With $d \geq 6$ the fifth term is at least $p + 24$. Trying $p = 5$ and $d = 6$ gives $5, 11, 17, 23, 29$, all prime, and no smaller fifth term is possible since $p \geq 5$ (the starts $p = 2$ and $p = 3$ fail as above). The answer is 29.

2. Consider the parallelogram with vertices $(10, 45)$, $(10, 114)$, $(28, 153)$, and $(28, 84)$. A line through the origin cuts this figure into two congruent polygons. The slope of the line is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

A parallelogram is symmetric under the 180° rotation about its center, so any line through the center cuts it into two pieces that are swapped by that rotation, hence congruent. The center is the midpoint of a diagonal:

$$\left(\frac{10+28}{2}, \frac{45+153}{2}\right) = (19, 99).$$

The line through the origin and $(19, 99)$ has slope $\frac{99}{19}$, and $\gcd(99, 19) = 1$, so $m + n = 99 + 19 = 118.$

3. Find the sum of all positive integers n for which $n^2 - 19n + 99$ is a perfect square.

Solution:

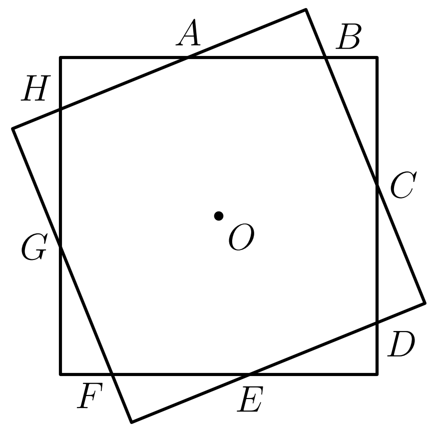
Suppose $n^2 - 19n + 99 = k^2$. Multiplying by 4 and completing the square gives $(2n - 19)^2 + 35 = (2k)^2$, so

$$\big(2k - (2n - 19)\big)\big(2k + (2n - 19)\big) = 35.$$

The two factors sum to $4k > 0$, so both are positive: the factor pairs are $(1, 35)$, $(5, 7)$, $(7, 5)$, and $(35, 1)$.

Subtracting the first factor from the second gives $2(2n - 19) = 34, 2, -2$, or -34, so $n = 18, 10, 9$, or 1. Each indeed makes the expression a perfect square ($81, 9, 9, 81$), and the sum is $18 + 10 + 9 + 1 = 38.$

4. The two squares shown share the same center O and have sides of length 1. The length of $\overline{AB}$ is $\frac{43}{99}$ and the area of octagon $ABCDEFGH$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution:

The whole configuration is unchanged by rotating 90° about O, which cycles the octagon side AB to CD, EF, GH, and it is also unchanged by the reflection that swaps the two squares, which carries those sides to BC, DE, FG, HA. So all eight sides of the octagon have the same length, $\frac{43}{99}$.

Segments from O to the eight vertices cut the octagon into 8 triangles. Each has base $\frac{43}{99}$ lying on a side of one of the unit squares, so its height from O is the distance from the center to that side, namely $\frac{1}{2}$. The area is

$$8 \cdot \frac{1}{2} \cdot \frac{43}{99} \cdot \frac{1}{2} = \frac{86}{99}.$$

Since $\gcd(86, 99) = 1$, the answer is $86 + 99 = 185$.

5. For any positive integer x, let $S(x)$ be the sum of the digits of x, and let $T(x)$ be $|S(x+2)-S(x)|$. For example, $T(199)=|S(201)-S(199)|=|3-19|=16.$ How many values $T(x)$ do not exceed 1999?

Solution:

If the last digit of x is at most 7, adding 2 changes no other digit, so $T(x)=2.$ Otherwise there is carrying. If x ends in the digit 8 preceded by exactly $m\geq 0$ nines, then $x+2$ replaces $\ldots a\underbrace{9\cdots 9}_{m}8$ by $\ldots(a{+}1)\underbrace{0\cdots 0}_{m}0$, so $S(x+2)-S(x)=1-9m-8$ and $T(x)=9m+7.$ If x ends in exactly $m\geq 1$ nines, then $x+2$ replaces $\ldots a\underbrace{9\cdots 9}_{m}$ by $\ldots(a{+}1)\underbrace{0\cdots 0}_{m-1}1$, so $S(x+2)-S(x)=2-9m$ and $T(x)=9m-2.$

Both carrying families give exactly the values $7,16,25,\ldots,$ that is, $9j+7$ for $j\geq 0,$ and every such value occurs. So the possible values of T are 2 together with all $9j+7.$ Requiring $9j+7\leq 1999$ gives $j\leq 221,$ which is 222 values, and $T=2$ adds one more, for a total of $223.$

6. A transformation of the first quadrant of the coordinate plane maps each point (x, y) to the point $(\sqrt{x}, \sqrt{y})$. The vertices of quadrilateral $ABCD$ are $A = (900, 300)$, $B = (1800, 600)$, $C = (600, 1800)$, and $D = (300, 900)$. Let k be the area of the region enclosed by the image of quadrilateral $ABCD$. Find the greatest integer that does not exceed k.

Solution:

Follow the four edges. Sides AB and DC lie on the lines $y = \frac{x}{3}$ and $y = 3x$, which map to the lines $v = \frac{u}{\sqrt{3}}$ and $v = \sqrt{3}\,u$ — rays from the origin at angles 30° and 60°. Sides AD and BC lie on $x + y = 1200$ and $x + y = 2400$, which map to arcs of the circles $u^2 + v^2 = 1200$ and $u^2 + v^2 = 2400$.

So the image is the part of the annulus between radii $\sqrt{1200}$ and $\sqrt{2400}$ lying between the 30° and 60° rays, one twelfth of the full annulus:

$$k = \frac{30}{360}\,\pi\,(2400 - 1200) = 100\pi \approx 314.16.$$

The greatest integer not exceeding k is 314.

7. There is a set of 1000 switches, each of which has four positions, called A, B, C, and D. When the position of any switch changes, it is only from A to B, from B to C, from C to D, or from D to A. Initially each switch is in position A. The switches are labeled with the 1000 different integers $2^x 3^y 5^z$, where x, y, and z take on the values $0, 1, \ldots, 9$. At step i of a 1000-step process, the ith switch is advanced one step, and so are all the other switches whose labels divide the label on the ith switch. After step 1000 has been completed, how many switches will be in position A?

Solution:

The switch labeled d is advanced exactly once for each step i whose label is a multiple of d. The multiples of $2^x 3^y 5^z$ among the labels are the $2^{x'} 3^{y'} 5^{z'}$ with $x \leq x' \leq 9$, etc., so that switch advances $(10-x)(10-y)(10-z)$ times. It returns to position A exactly when this count is a multiple of 4.

Write $a = 10 - x, b = 10 - y, c = 10 - z$, each ranging over 1 through 10. We count the triples where abc is *not* divisible by 4 : either all three are odd, or exactly one is even but not divisible by 4. Among $1, \ldots, 10$ there are 5 odd values and 3 values ($2, 6, 10$) that are twice an odd number. That gives $5^3 = 125$ triples of the first kind and $3 \cdot 3 \cdot 5^2 = 225$ of the second, or 350 in all.

Therefore $1000 - 350 = 650$ switches end in position A.

8. Let $\mathcal{T}$ be the set of ordered triples (x, y, z) of nonnegative real numbers that lie in the plane $x + y + z = 1$. Let us say that (x, y, z) supports (a, b, c) when exactly two of the following are true: $x \geq a, y \geq b, z \geq c$. Let $\mathcal{S}$ consist of those triples in $\mathcal{T}$ that support $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$. The area of $\mathcal{S}$ divided by the area of $\mathcal{T}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

$\mathcal{T}$ is the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Because $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1 = x + y + z$, whenever two of the inequalities $x \geq \frac{1}{2}, y \geq \frac{1}{3}, z \geq \frac{1}{6}$ hold, the third can hold only on a boundary segment of zero area. So $\mathcal{S}$ is, up to measure zero, the union of the three regions where a specific pair of inequalities holds.

The region with $x \geq \frac{1}{2}$ and $y \geq \frac{1}{3}$ becomes, after substituting $x = \frac{1}{2} + x'$ and $y = \frac{1}{3} + y'$, a copy of $\mathcal{T}$ with coordinate sum $1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, i.e. a triangle similar to $\mathcal{T}$ with ratio $\frac{1}{6}$ and area $\left(\frac{1}{6}\right)^2$ of $\mathcal{T}$. Likewise the pairs $\{x, z\}$ and $\{y, z\}$ give similar triangles with ratios $\frac{1}{3}$ and $\frac{1}{2}$.

The ratio of areas is

$$\frac{1}{36} + \frac{1}{9} + \frac{1}{4} = \frac{1 + 4 + 9}{36} = \frac{7}{18},$$

so $m + n = 7 + 18 = 25$.

9. A function f is defined on the complex numbers by $f(z) = (a + bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = \frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Solution:

The condition is $|f(z) - z| = |f(z)|$ for all z, that is, $|(a - 1 + bi)z| = |(a + bi)z|$. Dividing by $|z|$ (for $z \neq 0$) gives $|a - 1 + bi| = |a + bi|$, so

$$(a - 1)^2 + b^2 = a^2 + b^2,$$

which forces $a = \frac{1}{2}$.

Since $|a + bi| = 8$, we have $a^2 + b^2 = 64$, so $b^2 = 64 - \frac{1}{4} = \frac{255}{4}$. As $\gcd(255, 4) = 1$, the answer is $255 + 4 = 259$.

10. Ten points in the plane are given, with no three collinear. Four distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. The probability that some three of the segments form a triangle whose vertices are among the ten given points is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

There are $\binom{10}{2} = 45$ segments, so $\binom{45}{4} = 148995$ equally likely choices. Two distinct triangles share at most one edge, so together they use at least 5 segments; hence a set of 4 segments contains at most one triangle, and the favorable sets are counted exactly once by choosing a triangle and then a fourth segment:

$$\binom{10}{3} \cdot (45 - 3) = 120 \cdot 42 = 5040.$$

The probability is $\frac{5040}{148995} = \frac{16}{473}$, already in lowest terms ($473 = 11 \cdot 43$), so $m + n = 16 + 473 = 489$.

11. Given that $\sum_{k=1}^{35} \sin 5k = \tan \frac{m}{n}$, where angles are measured in degrees, and m and n are relatively prime positive integers that satisfy $\frac{m}{n} < 90$, find $m + n$.

Solution:

Multiply the sum by $2\sin 2.5^\circ$ and apply $2\sin 5k^\circ \sin 2.5^\circ = \cos(5k - 2.5)^\circ - \cos(5k + 2.5)^\circ$, so the sum telescopes:

$$2\sin 2.5^\circ \sum_{k=1}^{35} \sin 5k^\circ = \cos 2.5^\circ - \cos 177.5^\circ = 2\cos 2.5^\circ.$$

Hence the sum equals $\frac{\cos 2.5^\circ}{\sin 2.5^\circ} = \cot 2.5^\circ = \tan 87.5^\circ = \tan \frac{175}{2}^\circ$. Since $\gcd(175, 2) = 1$ and $\frac{175}{2} < 90$, we get $m + n = 175 + 2 = 177$.

12. The inscribed circle of triangle ABC is tangent to $\overline{AB}$ at P, and its radius is 21. Given that $AP = 23$ and $PB = 27$, find the perimeter of the triangle.

Solution:

Tangent segments from a point are equal, so the tangent lengths from A, B, C are 23, 27, and some z. Then the sides are $50, 23 + z, 27 + z$, the semiperimeter is $s = 50 + z$, and Heron's formula gives area $\sqrt{(50 + z) \cdot z \cdot 23 \cdot 27}$.

The area also equals $rs = 21(50 + z)$. Squaring $21(50 + z) = \sqrt{621z(50 + z)}$ and dividing by $50 + z$ gives $441(50 + z) = 621z$, so $180z = 22050$ and $z = \frac{245}{2}$.

The perimeter is $2s = 2\left(50 + \frac{245}{2}\right) = 345$.

13. Forty teams play a tournament in which every team plays every other team exactly once. No ties occur, and each team has a 50% chance of winning any game it plays. The probability that no two teams win the same number of games is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $\log_2 n$.

✓

Solution:

There are $\binom{40}{2} = 780$ games, hence 2^{780} equally likely outcomes. If all 40 win totals are distinct, they must be exactly $0, 1, \ldots, 39$. In that case the team with 39 wins beat everyone, the team with 38 wins beat everyone except that team, and so on: the assignment of totals to teams determines every game. Conversely each of the $40!$ assignments arises from exactly one outcome, so the probability is $\frac{40!}{2^{780}}$.

By Legendre's formula the power of 2 dividing $40!$ is $20 + 10 + 5 + 2 + 1 = 38$. In lowest terms the denominator is therefore $n = 2^{780-38} = 2^{742}$, so $\log_2 n = 742$.

14. Point P is located inside triangle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangle have lengths $AB = 13, BC = 14$, and $CA = 15$, and the tangent of angle PAB is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Let $\omega = \angle PAB = \angle PBC = \angle PCA$. In triangle ABP, the angles at A and B are ω and $B - \omega$, so $\angle APB = 180^\circ - B$ and the law of sines gives $BP = \frac{c \sin\omega}{\sin B}$. In triangle BCP, the angles at B and C are ω and $C - \omega$, so $\angle BPC = 180^\circ - C$ and $BP = \frac{a \sin(C-\omega)}{\sin C}$.

Equating and substituting $a = 2R\sin A, c = 2R\sin C$ yields $\sin^2 C \sin\omega = \sin A \sin B \sin(C - \omega)$. Expanding $\sin(C - \omega)$ and dividing by $\sin A \sin B \sin C \sin\omega$,

$$\frac{\sin C}{\sin A \sin B} = \cot\omega - \cot C,$$

and since $\sin C = \sin(A + B) = \sin A \cos B + \cos A \sin B$, the left side is $\cot A + \cot B$. Hence $\cot\omega = \cot A + \cot B + \cot C$.

Using $\cot A = \frac{b^2+c^2-a^2}{4K}$ and its analogues, where K is the area,

$$\cot\omega = \frac{a^2 + b^2 + c^2}{4K} = \frac{169 + 196 + 225}{4 \cdot 84} = \frac{590}{336} = \frac{295}{168},$$

since the 13-14-15 triangle has area 84. So $\tan\omega = \frac{168}{295}$, which is in lowest terms, and $m + n = 168 + 295 = 463.$

15. Consider the paper triangle whose vertices are $(0,0)$, $(34,0)$, and $(16,24)$. The vertices of its midpoint triangle are the midpoints of its sides. A triangular pyramid is formed by folding the triangle along the sides of its midpoint triangle. What is the volume of this pyramid?

Solution:

The midpoints are $M_1 = (17,0)$, $M_2 = (25,12)$, and $M_3 = (8,12)$. Folding the three corner triangles up along the sides of the midpoint triangle brings the corners together at one apex Q (each pair of glued half-sides has equal length). The apex keeps its folded distances: $QM_1 = 17$ (half of the side of length 34 that M_1 bisects), $QM_2 = 15$ (half of 30), and $QM_3 = 4\sqrt{13}$ (half of $\sqrt{16^2+24^2} = 8\sqrt{13}$).

Keep the midpoint triangle in the plane $z = 0$ and let $Q = (x,y,z)$. Subtracting $|Q - M_3|^2 = 208$ from $|Q - M_2|^2 = 225$ gives $(x-25)^2 - (x-8)^2 = 17$, so $x = 16$; subtracting $|Q - M_2|^2 = 225$ from $|Q - M_1|^2 = 289$ gives $2x + 3y = 68$, so $y = 12$. Then $z^2 = 289 - (16-17)^2 - 12^2 = 144$, so the apex is at height $z = 12$.

The base is the midpoint triangle, with area one quarter of the original triangle's $\frac{1}{2} \cdot 34 \cdot 24 = 408$, i.e. 102. The volume is

$$\frac{1}{3} \cdot 102 \cdot 12 = 408.$$

Problems: https://live.poshenloh.com/past-contests/aime/1999