1997 AIME Solutions

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1. How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?

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Solution:

Write $n = a^2 - b^2 = (a-b)(a+b)$. The factors $a-b$ and $a+b$ differ by the even number $2b$, so they have the same parity. If both are odd, n is odd; if both are even, $4 \mid n$. Hence no integer $n \equiv 2 \pmod{4}$ is a difference of two squares.

Conversely, every odd number $2k+1$ equals $(k+1)^2 - k^2$, and every multiple of 4, say $4k$, equals $(k+1)^2 - (k-1)^2$ (with $k - 1 \geq 0$ since $k \geq 1$).

Between 1 and 1000 there are 500 odd numbers and 250 multiples of 4, for a total of $500 + 250 = 750$.

2. The nine horizontal and nine vertical lines on an 8×8 checkerboard form r rectangles, of which s are squares. The number s/r can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

A rectangle is determined by choosing two of the nine horizontal lines and two of the nine vertical lines, so $r = \binom{9}{2}^2 = 36^2 = 1296.$

A $k \times k$ square can be placed in $(9-k)^2$ positions, so

$$s = \sum_{k=1}^{8} (9-k)^2 = 8^2 + 7^2 + \cdots + 1^2 = \frac{8 \cdot 9 \cdot 17}{6} = 204.$$

Then $\frac{s}{r} = \frac{204}{1296} = \frac{17}{108}$, which is in lowest terms, so $m + n = 17 + 108 = 125.$

3. Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?

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Solution:

Let a be the two-digit number and b the three-digit number. The condition is $1000a + b = 9ab$, which rearranges to $b(9a-1) = 1000a$. Since $\gcd(9a-1, a) = 1$, the number $9a-1$ must divide 1000.

For a two-digit a, $9a-1$ runs from 89 to 890, and $9a - 1 \equiv 8 \pmod{9}$. The only divisor of 1000 in that range congruent to 8 modulo 9 is 125, giving $a = 14$ and $b = \frac{1000 \cdot 14}{125} = 112$, which is indeed a three-digit number. Check: $14112 = 9 \cdot 14 \cdot 112$.

The requested sum is $14 + 112 = 126$.

4. Circles of radii $5, 5, 8,$ and $\frac{m}{n}$ are mutually externally tangent, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Let the radius-5 circles have centers P_1 and P_2, so $P_1P_2 = 5 + 5 = 10$, and let M be the midpoint. The radius-8 circle's center Q satisfies $QP_1 = QP_2 = 13$, so Q lies on the perpendicular bisector of $\overline{P_1P_2}$ at distance $\sqrt{13^2 - 5^2} = 12$ from M. Likewise the fourth circle, of radius r, has its center R on the same perpendicular bisector with $RP_1 = 5 + r$, so $RM = \sqrt{(5+r)^2 - 25} = \sqrt{r^2 + 10r}$.

The small circle nestles in the space between the other three, so R is between M and Q, and external tangency to the radius-8 circle gives

$$12 - \sqrt{r^2 + 10r} = 8 + r.$$

Then $\sqrt{r^2 + 10r} = 4 - r$, and squaring yields $r^2 + 10r = 16 - 8r + r^2$, so $18r = 16$ and $r = \frac{8}{9}$.

Thus $m + n = 8 + 9 = 17$.

5. The number r can be expressed as a four-place decimal $0.abcd,$ where $a, b, c,$ and d represent digits, any of which could be zero. It is desired to approximate r by a fraction whose numerator is 1 or 2 and whose denominator is an integer. The closest such fraction to r is $\frac{2}{7}$. What is the number of possible values for r?

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Solution:

Among fractions with numerator 1 or $2,$ the closest neighbors of $\frac{2}{7} \approx 0.2857$ are $\frac{1}{4} = 0.25$ below (note $\frac{2}{8} = \frac{1}{4}$) and $\frac{1}{3} \approx 0.3333$ above (note $\frac{2}{6} = \frac{1}{3}$); no other candidate lies between them. So $\frac{2}{7}$ is the unique closest fraction to r exactly when r is closer to $\frac{2}{7}$ than to both $\frac{1}{4}$ and $\frac{1}{3}$, i.e. when r lies strictly between the midpoints

$$\frac{1}{2}\left(\frac{1}{4}+\frac{2}{7}\right) = \frac{15}{56} = 0.26785\ldots \qquad \text{and} \qquad \frac{1}{2}\left(\frac{2}{7}+\frac{1}{3}\right) = \frac{13}{42} = 0.30952\ldots.$$

The four-place decimals in that interval are $0.2679, 0.2680, \ldots, 0.3095,$ and there are $3095 - 2679 + 1 = 417$ of them.

6. Point B is in the exterior of the regular n-sided polygon $A_1A_2\cdots A_n$, and A_1A_2B is an equilateral triangle. What is the largest value of n for which A_n, A_1, and B are consecutive vertices of a regular polygon?

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Solution:

Since B is outside the n-gon, the angles at A_1 — the interior angle $\angle A_nA_1A_2 = \frac{(n-2)180^\circ}{n}$, the equilateral angle $\angle A_2A_1B = 60^\circ$, and $\angle BA_1A_n$ — fill a full revolution, so

$$\angle BA_1A_n = 300^\circ - \frac{(n-2)180^\circ}{n} = 120^\circ + \frac{360^\circ}{n}.$$

Also $A_nA_1 = A_1B$, since both equal the side of the n-gon.

For A_n, A_1, B to be consecutive vertices of a regular m-gon, this angle must be the m-gon's interior angle: $120^\circ + \frac{360^\circ}{n} = 180^\circ - \frac{360^\circ}{m}$, which simplifies to $\frac{1}{m} = \frac{1}{6} - \frac{1}{n}$, so $m = \frac{6n}{n-6} = 6 + \frac{36}{n-6}$.

Thus $n - 6$ must divide 36, and the largest choice is $n - 6 = 36$, i.e. $n = 42$ (with $m = 7$).

7. A car travels due east at $\frac{2}{3}$ mile per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{1}{2}\sqrt{2}$ mile per minute. At time $t = 0,$ the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $\frac{1}{2}(t_1 + t_2).$

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Solution:

Put the car at the origin at $t = 0,$ with east as the positive x-direction and north as the positive y-direction. At time t the car is at $\left(\frac{2t}{3}, 0\right)$, and the storm center, moving southeast at speed $\frac{\sqrt{2}}{2}$ (components $\frac{1}{2}$ east and $\frac{1}{2}$ south), is at $\left(\frac{t}{2}, 110 - \frac{t}{2}\right)$.

The car is on the storm boundary when the squared distance is 51^2 :

$$\left(\frac{2t}{3} - \frac{t}{2}\right)^2 + \left(110 - \frac{t}{2}\right)^2 = 51^2,$$

that is $\frac{t^2}{36} + \frac{t^2}{4} - 110t + 12100 - 2601 = 0,$ or $\frac{5}{18}t^2 - 110t + 9499 = 0.$

The roots are t_1 and t_2, so by Vieta's formulas $t_1 + t_2 = \frac{110 \cdot 18}{5} = 396,$ and $\frac{1}{2}(t_1 + t_2) = 198.$

8. How many different 4×4 arrays whose entries are all 1's and -1's have the property that the sum of the entries in each row is 0 and the sum of the entries in each column is 0?

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Solution:

Each row must contain two 1's and two -1's, so identify each row with the pair of columns holding its 1's; each column must end up chosen by exactly two rows. There are $\binom{4}{2} = 6$ choices for row $1.$ Classify by how row 2 overlaps row $1.$

If row 2 uses the same pair (1 way), those two columns are full, so rows 3 and 4 must both use the complementary pair: 1 completion. If row 2 uses the complementary pair (1 way), every column has one 1 so far, so rows 3 and 4 need only be a complementary pair themselves: 6 choices for row $3,$ row 4 forced, giving 6 completions. If row 2 shares exactly one column with row 1 ($2 \cdot 2 = 4$ ways), one column is full, two have one $1,$ and one is empty; rows 3 and 4 must each take the empty column together with one of the two half-filled columns, so there are 2 completions.

The total is $6\,(1 \cdot 1 + 1 \cdot 6 + 4 \cdot 2) = 6 \cdot 15 = 90.$

9. Given a nonnegative real number x, let $\langle x\rangle$ denote the fractional part of x; that is, $\langle x\rangle = x - \lfloor x\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Suppose that a is positive, $\langle a^{-1}\rangle = \langle a^2\rangle$, and $2 < a^2 < 3$. Find the value of $a^{12} - 144a^{-1}$.

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Solution:

From $2 < a^2 < 3$ we get $\sqrt{2} < a < \sqrt{3}$, so $0 < a^{-1} < 1$ and $\langle a^{-1}\rangle = a^{-1}$, while $\langle a^2\rangle = a^2 - 2$. The condition becomes $a^{-1} = a^2 - 2$, i.e. $a^3 - 2a - 1 = 0$, which factors as

$$(a+1)(a^2 - a - 1) = 0.$$

Since $a > 0$, we get $a = \frac{1+\sqrt{5}}{2}$, the golden ratio, and indeed $a^2 = a + 1 \approx 2.618$ lies in $(2, 3)$.

Using $a^2 = a + 1$ repeatedly: $a^4 = (a+1)^2 = 3a + 2$, $a^8 = (3a+2)^2 = 9(a+1) + 12a + 4 = 21a + 13$, and $a^{12} = a^8 a^4 = (21a+13)(3a+2) = 63(a+1) + 81a + 26 = 144a + 89$. Also $a^{-1} = a - 1$ from $a^2 = a + 1$.

Therefore $a^{12} - 144a^{-1} = 144a + 89 - 144(a-1) = 89 + 144 = 233.$

10. Every card in a deck has a picture of one shape — circle, square, or triangle, which is painted in one of the three colors — red, blue, or green. Furthermore, each color is applied in one of three shades — light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:

- Either each of the three cards has a different shape or all three of the cards have the same shape.

- Either each of the three cards has a different color or all three of the cards have the same color.

- Either each of the three cards has a different shade or all three of the cards have the same shade.

How many different complementary three-card sets are there?

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Solution:

Given any two distinct cards, there is exactly one card completing them to a complementary set: in each attribute, if the two cards agree, the third card must share that value, and if they differ, the third must take the one remaining value. The completing card is distinct from both (the two given cards differ somewhere, and in that attribute the third card differs from each).

So the $\binom{27}{2} = 351$ pairs of cards each extend to one complementary set, and each complementary set is produced by $\binom{3}{2} = 3$ of these pairs. The number of sets is $\frac{351}{3} = 117.$

11. Let

$$x = \frac{\displaystyle\sum_{n=1}^{44} \cos n^\circ}{\displaystyle\sum_{n=1}^{44} \sin n^\circ}.$$

What is the greatest integer that does not exceed $100x$?

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Solution:

Multiply numerator and denominator by $2\sin\frac{1}{2}^\circ$. Since $2\cos n^\circ \sin\frac{1}{2}^\circ = \sin\left(n+\frac{1}{2}\right)^\circ - \sin\left(n-\frac{1}{2}\right)^\circ$ and $2\sin n^\circ \sin\frac{1}{2}^\circ = \cos\left(n-\frac{1}{2}\right)^\circ - \cos\left(n+\frac{1}{2}\right)^\circ$, both sums telescope:

$$x = \frac{\sin 44.5^\circ - \sin 0.5^\circ}{\cos 0.5^\circ - \cos 44.5^\circ} = \frac{2\cos 22.5^\circ \sin 22^\circ}{2\sin 22.5^\circ \sin 22^\circ} = \cot 22.5^\circ,$$

using the sum-to-product identities in the last step.

By the half-angle formula, $\cot 22.5^\circ = \frac{1+\cos 45^\circ}{\sin 45^\circ} = \sqrt{2}+1$. Hence $100x = 100\sqrt{2}+100 = 241.42\ldots$, and the greatest integer not exceeding it is 241.

12. The function f defined by $f(x) = \frac{ax+b}{cx+d}$, where $a, b, c,$ and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values except $\frac{-d}{c}$. Find the unique number that is not in the range of f.

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Solution:

Composing, $f(f(x)) = \frac{(a^2+bc)x+b(a+d)}{c(a+d)x+(bc+d^2)}$, and this equals x identically only if $c(a+d) = 0$. Since $c \neq 0$, we get $d = -a$, so $f(x) = \frac{ax+b}{cx-a}$.

A fixed point satisfies $cx^2 - ax = ax + b$, i.e. $cx^2 - 2ax - b = 0$, whose roots are 19 and 97. By Vieta's formulas, $19 + 97 = \frac{2a}{c}$, so $\frac{a}{c} = 58$.

Finally, y is in the range exactly when $y = \frac{ax+b}{cx-a}$ has a solution, i.e. $x(cy - a) = ay + b$. This solves for x unless $cy - a = 0$; and when $y = \frac{a}{c}$, the right side $\frac{a^2}{c} + b = \frac{a^2+bc}{c}$ is nonzero (otherwise f would be constant). So the unique number not in the range is $\frac{a}{c} = 58$.

13. Let S be the set of points in the Cartesian plane that satisfy

$$\Big|\big||x|-2\big|-1\Big|+\Big|\big||y|-2\big|-1\Big|=1.$$

If a model of S were built from wire of negligible thickness, then the total length of wire required would be $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a+b$.

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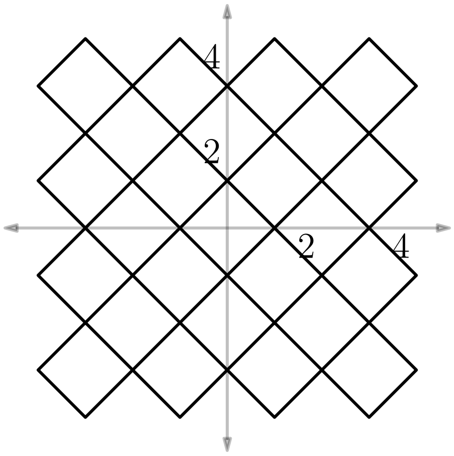
Solution:

Let $f(t)=\big|\,||t|-2|-1\,\big|$, so the equation is $f(x)+f(y)=1$. The function f is even, and for $t\geq 0:$ on $[0,2]$, $||t|-2|-1=(2-t)-1=1-t$, so $f(t)=|t-1|$; on $[2,4]$, $f(t)=|t-3|$; and for $t>4$, $f(t)=t-3>1$, which is too large. So on the relevant range, $f(t)=|t-a|$ where $a\in\{-3,-1,1,3\}$ is the nearest of those four values to t.

Therefore S is the union of the 16 taxicab circles

$$|x-a|+|y-b|=1,\qquad a,b\in\{-3,-1,1,3\},$$

which meet only at isolated points. Each is a square (diamond) with diagonal 2, hence side $\sqrt{2}$ and perimeter $4\sqrt{2}$.



The total length is $16\cdot 4\sqrt{2}=64\sqrt{2}$, so $a+b=64+2=66$.

14. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

By rotational symmetry we may fix v and let $w = ve^{2\pi ik/1997}$ with k uniform in $\{1, 2, \ldots, 1996\}$. Then

$$|v+w| = \left|1 + e^{2\pi ik/1997}\right| = 2\left|\cos\frac{\pi k}{1997}\right|.$$

Also $(2\cos 15^\circ)^2 = 2 + 2\cos 30^\circ = 2 + \sqrt{3}$, so the threshold is $\sqrt{2+\sqrt{3}} = 2\cos\frac{\pi}{12}$.

The condition $\left|\cos\frac{\pi k}{1997}\right| \ge \cos\frac{\pi}{12}$ holds exactly when $\frac{\pi k}{1997}$ is within $\frac{\pi}{12}$ of 0 or of π, i.e. $k \le \frac{1997}{12} = 166.41\ldots$ or $k \ge \frac{11\cdot 1997}{12} = 1830.58\ldots$. That gives $166 + 166 = 332$ favorable values of k.

The probability is $\frac{332}{1996} = \frac{83}{499}$, and 499 is prime, so $m + n = 83 + 499 = 582$.

15. The sides of rectangle $ABCD$ have lengths 10 and 11. An equilateral triangle is drawn so that no point of the triangle lies outside $ABCD$. The maximum possible area of such a triangle can be written in the form $p\sqrt{q}-r$, where p, q, and r are positive integers, and q is not divisible by the square of any prime number. Find $p+q+r$.

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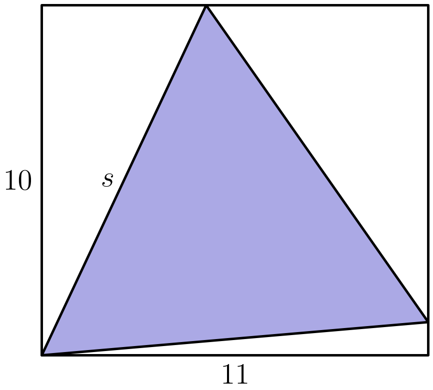
Solution:

Place the rectangle with corners $(0,0),(11,0),(11,10),(0,10)$. A maximal equilateral triangle can be enlarged unless it is pinned by the rectangle, and the extremal position has one vertex at a corner, say the origin, with the other two vertices $s(\cos\theta,\sin\theta)$ and $s\big(\cos(\theta+60^\circ),\sin(\theta+60^\circ)\big)$ touching the far sides $x=11$ and $y=10:$

$$s\cos\theta=11,\qquad s\sin(\theta+60^\circ)=10.$$



Dividing, $11\sin(\theta+60^\circ)=10\cos\theta$, and expanding the left side gives $\frac{11}{2}\sin\theta+\frac{11\sqrt{3}}{2}\cos\theta=10\cos\theta$, so $\tan\theta=\frac{20-11\sqrt{3}}{11}$ (about 4.9°, a legal tilt). Then

$$s^2=\frac{121}{\cos^2\theta}=121\left(1+\tan^2\theta\right)=121+\left(20-11\sqrt{3}\right)^2=884-440\sqrt{3}.$$

The area is $\frac{\sqrt{3}}{4}s^2=\frac{\sqrt{3}}{4}\left(884-440\sqrt{3}\right)=221\sqrt{3}-330\approx 52.8$, which indeed beats the untilted triangle of side 10. Thus $p+q+r=221+3+330=554.$

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