1996 AIME Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/aime/1996/solutions

1. In a magic square, the sum of the three entries in any row, column, or diagonal is the same value. The figure shows four of the entries of a magic square. Find x.

x	19	96
1		

Solution:

Let the center entry be e and the common sum be S. In any 3×3 magic square, $S = 3e$ and entries opposite across the center sum to $2e$. Thus the bottom-left entry is $2e - 96$. The first column and first row give

$$\begin{aligned} x + 1 + (2e - 96) &= 3e, \\ x + 19 + 96 &= 3e. \end{aligned}$$

The first equation says $e = x - 95$, while the second says $3e = x + 115$. Hence $3x - 285 = x + 115$, so $x = 200$.

2. For each real number x, let $\lfloor x \rfloor$ denote the greatest integer that does not exceed x. For how many positive integers n is it true that $n < 1000$ and that $\lfloor \log_2 n \rfloor$ is a positive even integer?

Solution:

If $\lfloor \log_2 n \rfloor = k$, then $2^k \leq n < 2^{k+1}$. The positive even possibilities are $k = 2, 4, 6, 8,$ since $2^{10} > 1000$. The corresponding interval sizes are $2^2, 2^4, 2^6$, and 2^8. Therefore the requested number is

$$4 + 16 + 64 + 256 = 340.$$

3. Find the smallest positive integer n for which the expansion of $(xy - 3x + 7y - 21)^n$, after like terms have been collected, has at least 1996 terms.

Solution:

The base factors as

$$\begin{aligned} &xy - 3x + 7y - 21 \\ &= (x + 7)(y - 3). \end{aligned}$$

Hence its nth power is $(x + 7)^n (y - 3)^n$. Each exponent of x from 0 through n can occur with each exponent of y from 0 through n, and every resulting coefficient is nonzero. Thus there are $(n + 1)^2$ terms. Since $44^2 < 1996 \leq 45^2$, the least possible $n + 1$ is 45, so $n = 44$.

4. A wooden cube, whose edges are one centimeter long, rests on a horizontal surface. Illuminated by a point source of light that is x centimeters directly above an upper vertex, the cube casts a shadow on the horizontal surface. The area of the shadow, which does not include the area beneath the cube, is 48 square centimeters. Find the greatest integer that does not exceed $1000x$.

Solution:

The light is $x+1$ centimeters above the surface and x centimeters above the cube's top face. By similarity, the projection of that unit-square face is a square of side $\frac{x+1}{x}$. This square contains the unit-square area beneath the cube, so

$$\left(\frac{x+1}{x}\right)^2 - 1 = 48.$$

Therefore $\frac{x+1}{x} = 7$, giving $x = \frac{1}{6}$. The greatest integer not exceeding $1000x = 166\frac{2}{3}$ is 166.

5. Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are $a, b,$ and $c,$ and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b, b + c,$ and $c + a.$ Find $t.$

Solution:

Vieta's formulas give

$$\begin{aligned} a + b + c &= -3, \\ ab + bc + ca &= 4, \\ abc &= 11. \end{aligned}$$

Also,

$$\begin{gathered} (a+b)(b+c)(c+a) \\ = (a+b+c)(ab+bc+ca) \\ -abc = -23. \end{gathered}$$

This is the product of the roots of the second monic cubic, so its constant term is the negative of that product. Hence $t = 23.$

6. In a five-team tournament, each team plays one game with every other team. Each team has a 50% chance of winning any game it plays. There are no ties. Let $\frac{m}{n}$ be the probability that the tournament will produce neither an undefeated team nor a winless team, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

There are $2^{10}=1024$ outcomes. A specified undefeated team forces its four games and leaves the other six arbitrary, so there are $5\cdot 2^6=320$ outcomes with an undefeated team. The same count holds for a winless team. If distinct specified teams are undefeated and winless, seven games are forced and the three games among the other teams are arbitrary. Thus the intersection count is $5\cdot 4\cdot 2^3=160$. By inclusion-exclusion, the desired count is

$$1024-320-320+160=544.$$

The probability is $\frac{544}{1024}=\frac{17}{32}$, so $m+n=49$.

7. Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

Solution:

Under the identity rotation, all $\binom{49}{2} = 1176$ pairs are fixed. A quarter-turn or three-quarter-turn has only orbits of sizes 1 and 4, so it fixes no two-square set. A half-turn fixes exactly the 24 pairs of squares opposite one another across the center. Burnside's Lemma therefore gives

$$\frac{1176 + 0 + 24 + 0}{4} = 300$$

inequivalent colorings.

8. The harmonic mean of two positive numbers is the reciprocal of the arithmetic mean of their reciprocals. For how many ordered pairs of positive integers (x, y), with $x < y$, is the harmonic mean of x and y equal to 6^{20}?

Solution:

Let $N = 6^{20}$. Rearranging the harmonic-mean equation gives

$$(2x - N)(2y - N) = N^2.$$

Because $x < N < y$ and $N < 2x$, the two factors are positive, and both must be even. Conversely, each factorization $AB = N^2$ with even $A < B$ gives one valid pair via $x = \frac{A+N}{2}$ and $y = \frac{B+N}{2}$.

Now $N^2 = 2^{40}3^{40}$. For both complementary factors to be even, the exponent of 2 in A can be $1, \ldots, 39$, while the exponent of 3 can be $0, \ldots, 40$. This gives $39 \cdot 41 = 1599$ divisors A, including the central factor $A = N$. Pairing complementary divisors and excluding that central case gives $\frac{1599-1}{2} = 799$.

9. A bored student walks down a hall that contains a row of closed lockers, numbered 1 to $1024.$ He opens locker $1,$ and then alternates between skipping and opening each closed locker thereafter. When he reaches the end of the hall, the student turns around and starts back. He opens the first closed locker he encounters, and then alternates between skipping and opening each closed locker thereafter. The student continues wandering back and forth in this manner until every locker is open. What is the number of the last locker he opens?

Solution:

On every trip the student opens the first, third, fifth, and so on among the remaining lockers in his direction of travel. Tracking the closed arithmetic sequence after each trip gives

trip	first	difference	count
1	2	2	512
2	2	4	256
3	6	8	128
4	6	16	64
5	22	32	32
6	22	64	16
7	86	128	8
8	86	256	4
9	342	512	2

Thus only lockers 342 and 854 remain after the ninth trip. On the tenth trip, starting from the right, locker 854 is opened and 342 remains. Therefore the last locker opened is $342.$

10. Find the smallest positive integer solution to

$$\tan(19x^\circ) = \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}.$$

Solution:

The tangent addition formula gives

$$\begin{gathered}\frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ} \\ = \tan(45^\circ + 96^\circ) \\ = \tan 141^\circ.\end{gathered}$$

Hence $19x \equiv 141 \pmod{180}$. Since $19^2 = 361 \equiv 1 \pmod{180}$, multiplying by 19 gives

$$x \equiv 19 \cdot 141 \equiv 159 \pmod{180}.$$

The smallest positive solution is 159.

11. Let P be the product of the roots of

$$z^6 + z^4 + z^3 + z^2 + 1 = 0$$

that have positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $r > 0$ and $0 \leq \theta < 360$. Find θ.

Solution:

No root is zero. Dividing by z^3 and setting $w = z + z^{-1}$ gives

$$\begin{aligned} w^3 - 2w + 1 &= 0, \\ (w-1)(w^2 + w - 1) &= 0. \end{aligned}$$

Its three roots are

$$\begin{aligned} 1 &= 2\cos 60^\circ, \\ \frac{\sqrt{5}-1}{2} &= 2\cos 72^\circ, \\ -\frac{1+\sqrt{5}}{2} &= 2\cos 144^\circ. \end{aligned}$$

For each value $w = 2\cos\phi$, the corresponding roots of the original equation are $e^{i\phi}$ and $e^{-i\phi}$. Thus the roots with positive imaginary part have arguments 60°, 72°, and 144°. Their product has argument $60 + 72 + 144 = 276^\circ$, so $\theta = 276$.

12. For each permutation $a_1, a_2, a_3, \ldots, a_{10}$ of the integers $1, 2, 3, \ldots, 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4|$$
$$+|a_5 - a_6| + |a_7 - a_8|$$
$$+|a_9 - a_{10}|.$$

The average value of all such sums can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:

Each paired difference has the same expected value as the difference of a uniformly selected unordered pair from $1, \ldots, 10$. Therefore

$$\begin{aligned}\mathbb{E}|a_1 - a_2| &= \frac{\sum_{d=1}^{9} d(10-d)}{\binom{10}{2}} \\ &= \frac{10\sum_{d=1}^{9} d}{45} \\ &\quad - \frac{\sum_{d=1}^{9} d^2}{45} \\ &= \frac{450 - 285}{45} = \frac{11}{3}.\end{aligned}$$

By linearity of expectation, the average of the five-term sum is $\frac{5 \cdot 11}{3} = \frac{55}{3}$. Thus $p + q = 58$.

13. In triangle $ABC, AB = \sqrt{30}, AC = \sqrt{6}$, and $BC = \sqrt{15}$. There is a point D for which $\overline{AD}$ bisects $\overline{BC}$, and $\angle ADB$ is a right angle. The ratio

$$\frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ABC)}$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Let E be the midpoint of $\overline{BC}$. Then A, E, D are collinear. The median formula gives

$$\begin{aligned} AE^2 &= \frac{2AB^2 + 2AC^2 - BC^2}{4} \\ &= \frac{57}{4}. \end{aligned}$$

Since $AB^2 > AE^2 + BE^2$, angle AEB is obtuse, so the perpendicular foot D lies beyond E. Both $\triangle ABD$ and $\triangle EBD$ are right at D. Therefore

$$\begin{gathered} AB^2 - BE^2 \\ = (AE + DE)^2 - DE^2 \\ = AE^2 + 2AE \cdot DE. \end{gathered}$$

Substitution gives $30 - \frac{15}{4} = \frac{57}{4} + 2AE \cdot DE$, so $\frac{DE}{AE} = \frac{8}{19}$.

Triangles ABE and DBE have bases AE and DE on the same line and share the altitude from B, while $[ABC] = 2[ABE]$. Hence

$$
\begin{aligned}
\frac{[ADB]}{[ABC]} &= \frac{[ABE]+[DBE]}{2[ABE]} \\
&= \frac{1}{2}\left(1+\frac{8}{19}\right) \\
&= \frac{27}{38}.
\end{aligned}
$$

Thus $m+n=65$.

14. A $150 \times 324 \times 375$ rectangular solid is made by gluing together $1 \times 1 \times 1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1 \times 1 \times 1$ cubes?

Solution:

For an $a \times b \times c$ array, the diagonal crosses $a-1, b-1,$ and $c-1$ internal grid planes of the three orientations. Crossings of two orientations coincide $\gcd(a,b)-1, \gcd(a,c)-1,$ and $\gcd(b,c)-1$ times, and triple crossings occur $\gcd(a,b,c)-1$ times. Adding one for the initial cube and applying inclusion-exclusion gives

$$
\begin{gathered}
a+b+c-\gcd(a,b) \\
-\gcd(a,c)-\gcd(b,c) \\
+\gcd(a,b,c).
\end{gathered}
$$

Here the pairwise gcds are $6, 75, 3,$ and the triple gcd is 3. Thus the number of cube interiors met is

$$
\begin{gathered}
150+324+375 \\
-6-75-3+3=768.
\end{gathered}
$$

15. In parallelogram $ABCD$, let O be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Angles CAB and DBC are each twice as large as angle DBA, and angle ACB is r times as large as angle AOB. Find the greatest integer that does not exceed $1000r$.

Solution:

Let $\angle DBA = \alpha$. Then $\angle DBC = \angle CAB = 2\alpha$, so in $\triangle ABC$ the angles are $2\alpha, 3\alpha,$ and $180^\circ - 5\alpha$. Because $AD \parallel BC$, triangle ABD has angles $\alpha, 2\alpha,$ and $180^\circ - 3\alpha$.

Applying the law of sines in the two triangles and using $AD = BC$ gives

$$\frac{AB}{BC} = \frac{\sin 5\alpha}{\sin 2\alpha} = \frac{\sin 2\alpha}{\sin \alpha}.$$

With $u = \cos^2 \alpha$, this becomes

$$16u^2 - 16u + 1 = 0.$$

Since $5\alpha < 180^\circ$, the valid root is $u = \frac{2+\sqrt{3}}{4} = \cos^2 15^\circ$, so $\alpha = 15^\circ$. Thus $\angle ACB = 105^\circ$. In $\triangle AOB$, the angles at A and B are 30° and 15°, so $\angle AOB = 135^\circ$.
Therefore

$$r = \frac{105}{135} = \frac{7}{9},$$
$$\lfloor 1000r \rfloor = 777.$$

Problems: https://live.poshenloh.com/past-contests/aime/1996