1995 AIME Solutions

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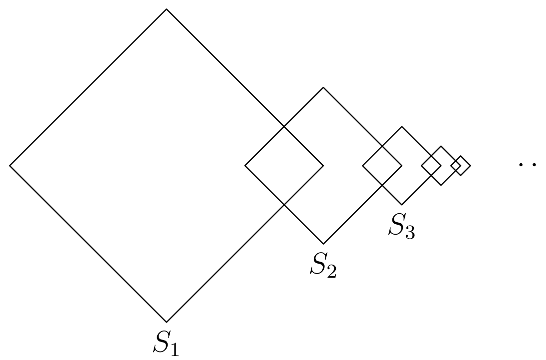
1. Square S_1 is $1 \times 1.$ For $i \geq 1,$ the lengths of the sides of square S_{i+1} are half the lengths of the sides of square $S_i,$ two adjacent sides of square S_i are perpendicular bisectors of two adjacent sides of square $S_{i+1},$ and the other two sides of square S_{i+1} are the perpendicular bisectors of two adjacent sides of square $S_{i+2}.$ The total area enclosed by at least one of S_1, S_2, S_3, S_4, S_5 can be written in the form $\frac{m}{n},$ where m and n are relatively prime positive integers. Find $m-n.$



Solution:

The sum of the five square areas is

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$$
$$= \frac{1364}{1024}.$$

The perpendicular-bisector placement makes the overlap of each adjacent pair one fourth of the smaller square. These four overlaps are disjoint and have total area

$$\frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024}$$
$$= \frac{85}{1024}.$$

Thus the union has area $\frac{1364-85}{1024} = \frac{1279}{1024}$, and $m-n = 1279-1024 = 255.$

2. Find the last three digits of the product of the positive roots of

$$\sqrt{1995}\,x^{\log_{1995} x} = x^2.$$

Solution:

Put $y = \log_{1995} x$, so $x = 1995^y$. The equation becomes

$$1995^{\frac{1}{2}+y^2} = 1995^{2y},$$

and hence $(y-1)^2 = \frac{1}{2}$. The two values of y have sum 2, so the product of the corresponding positive roots is 1995^2. Since $1995 \equiv -5 \pmod{1000}$, its last three digits are 025. The requested AIME answer is 25.

3. Starting at $(0,0)$, an object moves in the coordinate plane via a sequence of steps, each of length one. Each step is left, right, up, or down, all four equally likely. Let p be the probability that the object reaches $(2,2)$ in six or fewer steps. Given that p can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Solution:

There are $\binom{4}{2} = 6$ four-step paths to $(2,2)$. There are $60 + 60 = 120$ six-step paths ending there: the extra opposite pair is either left-right or down-up. Of these, $6 \cdot 4 = 24$ first reach the target at step 4 and then make a two-step return. Therefore

$$p = \frac{6}{4^4} + \frac{120-24}{4^6} = \frac{3}{64}.$$

Thus $m + n = 3 + 64 = 67$.

4. Circles of radius 3 and 6 are externally tangent to each other and are internally tangent to a circle of radius $9.$ The circle of radius 9 has a chord that is a common external tangent of the other two circles. Find the square of the length of this chord.

Solution:

Put the radius-9 circle at the origin and the smaller centers at $(-6, 0)$ and $(3, 0)$. Write the common external tangent as $\mathbf{n} \cdot (x, y) = c$, where $\mathbf{n}$ is a unit normal. Its signed distances from the two centers differ by $6 - 3 = 3$, so $9n_x = 3$ and $n_x = \frac{1}{3}$. Using the distance 3 from $(-6, 0)$ gives $|c| = 5$. Thus the chord lies 5 units from the large center, and its squared length is

$$4(9^2 - 5^2) = 4 \cdot 56 = 224.$$

5. For certain real values of $a, b, c,$ and $d,$ the equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i,$ where $i = \sqrt{-1}.$ Find $b.$

Solution:

Let the first two roots be α, β. Since $\alpha\beta = 13 + i$ is not real, they are not conjugates, so the other roots are $\overline{\alpha}, \overline{\beta}$. Hence $\alpha + \beta = 3 - 4i$ and $\overline{\alpha}\,\overline{\beta} = 13 - i$. By Vieta's formulas,

$$\begin{aligned} b &= \alpha\beta + \overline{\alpha}\,\overline{\beta} \\ &\quad + (3 - 4i)(3 + 4i) \\ &= 26 + 25 = 51. \end{aligned}$$

6. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

Solution:

The number $n^2 = 2^{62}3^{38}$ has $63 \cdot 39 = 2457$ divisors. Pairing d with $\frac{n^2}{d}$ leaves only n unpaired, so $\frac{2457-1}{2} = 1228$ divisors lie below n. The number n has $32 \cdot 20 = 640$ divisors, of which 639 are below n. Therefore the requested count is $1228 - 639 = 589$.

7. Given that

$$(1+\sin t)(1+\cos t) = \frac{5}{4}$$

and

$$\begin{aligned}(1-\sin t)(1-\cos t)\\ &= \frac{m}{n} - \sqrt{k},\end{aligned}$$

where $k, m,$ and n are positive integers with m and n relatively prime, find $k+m+n$.

Solution:

Let $u = \sin t + \cos t$. Since $\sin t \cos t = \frac{u^2-1}{2}$,

$$\begin{aligned}(1+\sin t)(1+\cos t)\\ &= \frac{(u+1)^2}{2} = \frac{5}{4}.\end{aligned}$$

The feasible sign gives $u+1 = \sqrt{\frac{5}{2}}$. Therefore

$$\begin{aligned}(1-\sin t)(1-\cos t)\\ &= \frac{(1-u)^2}{2}\\ &= \frac{13}{4} - \sqrt{10}.\end{aligned}$$

Thus $k+m+n = 10+13+4 = 27$.

8. For how many ordered pairs of positive integers (x, y), with $y < x \leq 100$, are both $\frac{x}{y}$ and $\frac{x+1}{y+1}$ integers?

Solution:

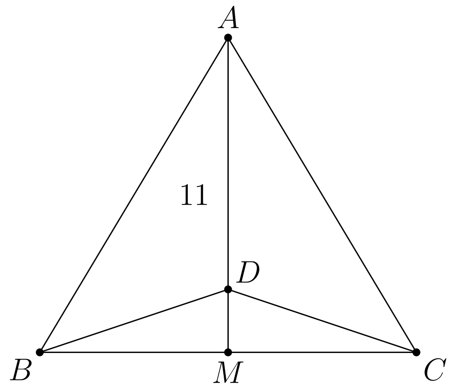
Write $x = ay$. Modulo $y + 1$, we have $y \equiv -1$, so $y + 1 \mid x + 1$ exactly when $a \equiv 1 \pmod{y+1}$. Since $x > y$, write $a = 1 + t(y + 1)$ with $t \geq 1$. The bound $x \leq 100$ becomes

$$t \leq \left\lfloor \frac{100 - y}{y(y+1)} \right\rfloor .$$

Only $1 \leq y \leq 9$ contribute, giving

$$\begin{aligned} 49 + 16 + 8 + 4 + 3& \\ &+ 2 + 1 + 1 + 1 \\ &= 85. \end{aligned}$$

9. Triangle ABC is isosceles, with $AB = AC$ and altitude $AM = 11$. Suppose that there is a point D on $\overline{AM}$ with $AD = 10$ and $\angle BDC = 3\angle BAC$. Then the perimeter of $\triangle ABC$ may be written in the form $a + \sqrt{b}$, where a and b are integers. Find $a + b$.



Solution:

Let $BM = x$ and $\alpha = \angle BAM$, so $\tan\alpha = \frac{x}{11}$. Because $DM = AM - AD = 1$, symmetry gives $\angle BDC = 2\arctan x$, while $\angle BAC = 2\alpha$. Hence $\arctan x = 3\alpha$. Put $t = \tan\alpha = \frac{x}{11}$. Then

$$\frac{3t - t^3}{1 - 3t^2} = 11t,$$

so $t^2 = \frac{1}{4}$ and $x = \frac{11}{2}$. Thus $BC = 11$ and $AB = \frac{11\sqrt{5}}{2}$, making the perimeter

$$11 + 11\sqrt{5} = 11 + \sqrt{605}.$$

Therefore $a + b = 11 + 605 = 616.$

10. What is the largest positive integer that is not the sum of a positive integral multiple of 42 and a positive composite integer?

Solution:

A number in residue class $r \pmod{42}$ becomes representable immediately after the first positive composite in that class. Composite residues r themselves supply that first value. For the remaining residues, suitable first composites are

$$\begin{aligned}
&0:42,\ 1:85,\ 2:44,\\
&3:45,\ 5:215,\ 7:49,\\
&11:95,\ 13:55,\ 17:143,\\
&19:145,\ 23:65,\ 29:155,\\
&31:115,\ 37:121,\ 41:125.
\end{aligned}$$

The largest entry is 215, and $215-42, 215-84, 215-126$, and $215-168$ are all prime. Hence 215 is not representable, while every larger integer is.

11. A right rectangular prism P (i.e., a rectangular parallelepiped) has sides of integral length a, b, c, with $a \le b \le c$. A plane parallel to one of the faces of P cuts P into two prisms, one of which is similar to P, and both of which have nonzero volume. Given that $b = 1995$, for how many ordered triples (a, b, c) does such a plane exist?

Solution:

Let the similar smaller prism have sorted sides $x \le y \le z$. It shares two side lengths with P, and all three of its sorted sides are smaller than the corresponding sides of P. The only possible matching is $y = a,\ z = b = 1995$. Similarity then gives

$$\frac{x}{a} = \frac{a}{1995} = \frac{1995}{c},$$

so $ac = 1995^2$. Conversely every factor pair $a < c$ gives a nondegenerate cut. Since $1995 = 3 \cdot 5 \cdot 7 \cdot 19$, its square has $3^4 = 81$ divisors. Excluding the central pair $a = c = 1995$ and taking one divisor from each remaining pair gives $\frac{81-1}{2} = 40$.

12. Pyramid $OABCD$ has square base $ABCD$, congruent edges $\overline{OA}, \overline{OB}, \overline{OC}$, and $\overline{OD}$, and $\angle AOB = 45^\circ$. Let θ be the measure of the dihedral angle formed by faces OAB and OBC. Given that $\cos\theta = m + \sqrt{n}$, where m and n are integers, find $m + n$.

Solution:

Take adjacent base vertices $A = (1,1,0), B = (-1,1,0), C = (-1,-1,0)$ and $O = (0,0,h)$. From $\angle AOB = 45^\circ$,

$$\frac{h^2}{h^2+2} = \frac{1}{\sqrt{2}},$$
$$h^2 = 2 + 2\sqrt{2}.$$

Normals to the two faces may be taken as $(0, 2h, 2)$ and $(-2h, 0, 2)$. Their acute angle has cosine $\frac{1}{h^2+1} = 3 - 2\sqrt{2}$. The interior dihedral angle is its supplement, so

$$\cos\theta = 2\sqrt{2} - 3 = -3 + \sqrt{8}.$$

Thus $m + n = -3 + 8 = 5.$

13. Let $f(n)$ be the integer closest to $\sqrt[4]{n}$. Find

$$\sum_{k=1}^{1995} \frac{1}{f(k)}.$$

Solution:

For $j \geq 1$, the number of integers n closest to j^4 in fourth-root scale is

$$\begin{aligned} &\left(j+\frac{1}{2}\right)^4-\left(j-\frac{1}{2}\right)^4 \\ &= 4j^3+j. \end{aligned}$$

For $j = 1, \ldots, 6$, these account for

$$\sum_{j=1}^{6}(4j^3+j) = 1785$$

values, and their contribution to the requested sum is

$$\begin{aligned} \sum_{j=1}^{6} \frac{4j^3+j}{j} &= 4\sum_{j=1}^{6} j^2+6 \\ &= 370. \end{aligned}$$

The remaining $1995-1785=210$ values have $f(n)=7$, contributing 30. The total is $400.$

14. In a circle of radius $42,$ two chords of length 78 intersect at a point whose distance from the center is $18.$ The two chords divide the interior of the circle into four regions. Two of these regions are bordered by segments of unequal lengths, and the area of either of them can be expressed uniquely in the form $m\pi - n\sqrt{d},$ where $m, n,$ and d are positive integers and d is not divisible by the square of any prime. Find $m + n + d.$

Solution:

Put the center at $O = (0, 0)$ and the intersection at $P = (18, 0).$ A length-78 chord is $9\sqrt{3}$ from $O,$ so a line through P containing such a chord makes angle 60° or 120° with $OP.$ Solving along either line gives segment lengths 30 and $48.$

For either region bordered by unequal segments, the two arc endpoints subtend 60° at $O.$ Its area is the sector minus $\triangle OAB$ plus $\triangle PAB$:

$$\begin{aligned}\frac{60}{360}\pi(42)^2 - \frac{1}{2}(42)^2 \sin 60^\circ \\ &+ \frac{1}{2}(30)(48)\sin 60^\circ \\ &= 294\pi - 81\sqrt{3}.\end{aligned}$$

Hence $m + n + d = 378.$

15. Let p be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Given that p can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Solution:

Let q_i be the success probability with i consecutive heads and no trailing tail, and let t be the probability after one tail. Then

$$\begin{aligned} q_i &= \frac{q_{i+1} + t}{2} \quad (0 \le i < 4), \\ q_4 &= \frac{1 + t}{2}, \\ t &= \frac{q_1}{2}. \end{aligned}$$

Working backward gives $q_1 = \frac{1+15t}{16}$. Since $q_1 = 2t$, we get $t = \frac{1}{17}$ and $q_1 = \frac{2}{17}$. Therefore

$$p = q_0 = \frac{q_1 + t}{2} = \frac{3}{34},$$

so $m + n = 3 + 34 = 37$.

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