1994 AIME Solutions

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1. The increasing sequence $3, 15, 24, 48, \ldots$ consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994th term of the sequence is divided by 1000?

Solution:

The terms are $k^2 - 1$ for integers $k \geq 2$ not divisible by 3. In each block of three consecutive k's there are two eligible values. The $1994 = 2 \cdot 997$th corresponds to $k = 3(997) + 1 = 2992$. Since $2992 \equiv 992 \equiv -8 \pmod{1000}$,

$$\begin{aligned} k^2 - 1 &\equiv (-8)^2 - 1 \\ &\equiv 63 \pmod{1000}. \end{aligned}$$

2. A circle with diameter $\overline{PQ}$ of length 10 is internally tangent at P to a circle of radius 20. Square $ABCD$ is constructed with A and B on the larger circle, $\overline{CD}$ tangent at Q to the smaller circle, and the smaller circle outside $ABCD$. The length of $\overline{AB}$ can be written in the form $m + \sqrt{n}$, where m and n are integers. Find $m + n$.

Solution:

Put the large circle at the origin with $P = (20, 0)$ and $Q = (10, 0)$. Let the square's side be s. Because the smaller circle is outside the square, $\overline{CD}$ lies on $x = 10$ and the parallel chord $\overline{AB}$ lies on $x = 10 - s$. A chord of the radius-20 circle then gives

$$s = 2\sqrt{400 - (10 - s)^2}.$$

Squaring yields $s^2 - 16s - 240 = 0$, so $s = 8 + 4\sqrt{19} = 8 + \sqrt{304}$. Thus $m + n = 8 + 304 = 312$.

3. The function f has the property that, for each real number x,

$$f(x) + f(x-1) = x^2.$$

If $f(19) = 94$, what is the remainder when $f(94)$ is divided by 1000?

Solution:

First $f(20) = 20^2 - f(19) = 306$. Subtracting through two steps gives

$$\begin{aligned} f(x) &= x^2 - (x-1)^2 + f(x-2) \\ &= 2x - 1 + f(x-2). \end{aligned}$$

Therefore

$$\begin{aligned} f(94) &= 306 + \sum_{j=11}^{47} (4j - 1) \\ &= 306 + 4(1073) - 37 \\ &= 4561. \end{aligned}$$

The requested remainder is 561.

4. Find the positive integer n for which

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor \\ + \cdots + \lfloor \log_2 n \rfloor = 1994.$$

(For real x, $\lfloor x \rfloor$ is the greatest integer $\leq x$.)

Solution:

For $2^j \leq k < 2^{j+1}$, the summand is j. Thus the sum through $255 = 2^8 - 1$ is

$$\sum_{j=0}^{7} j 2^j = (8-2)2^8 + 2 = 1538.$$

The remaining $1994 - 1538 = 456$ is $57 \cdot 8$, so we include 57 more integers beginning with 256. Hence $n = 255 + 57 = 312$.

5. Given a positive integer n, let $p(n)$ be the product of the nonzero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$\begin{aligned} S = p(1) + p(2) + p(3) \\ + \cdots + p(999). \end{aligned}$$

What is the largest prime factor of S?

Solution:

For one digit position, the sum of its effective factors is $1 + 1 + 2 + \cdots + 9 = 46$, where the first 1 represents digit 0. Thus the sum over 000 through 999 is 46^3. Removing the artificial contribution 1 from 000 gives

$$\begin{aligned} S &= 46^3 - 1 \\ &= 45(46^2 + 46 + 1) \\ &= 45 \cdot 2163. \end{aligned}$$

Since $2163 = 3 \cdot 721 = 3 \cdot 7 \cdot 103$, the largest prime factor is 103.

6. The graphs of the equations

$$y = k,$$
$$y = \sqrt{3}x + 2k,$$
$$y = -\sqrt{3}x + 2k,$$

are drawn in the coordinate plane for $k = -10, -9, -8, \ldots, 9, 10$. These 63 lines cut part of the plane into equilateral triangles of side $\frac{2}{\sqrt{3}}$. How many such triangles are formed?

Solution:

A unit triangular cell is determined by indices $i, j, k \in [-10, 10]$ satisfying $i = j + k \pm 1$. For the plus sign, $-11 \leq j + k \leq 9$. There are $21 - |s|$ ordered pairs (j, k) with sum s, so this orientation contributes

$$\sum_{s=-11}^{9} (21 - |s|) = 330.$$

By symmetry the other orientation also contributes 330, for a total of 660.

7. For certain ordered pairs (a, b) of real numbers, the system of equations

$$\begin{aligned} ax + by &= 1, \\ x^2 + y^2 &= 50 \end{aligned}$$

has at least one solution, and each solution is an ordered pair (x, y) of integers. How many such ordered pairs (a, b) are there?

Solution:

The circle has the 12 lattice points obtained from $(\pm 1, \pm 7), (\pm 5, \pm 5), (\pm 7, \pm 1)$. A secant satisfying the condition is determined by any two non-antipodal lattice points. This gives $\binom{12}{2} - 6 = 60$ lines; antipodal pairs are excluded because their line passes through the origin and cannot have equation $ax + by = 1$. There are also 12 tangents, one at each lattice point. Each line has a unique normalization $ax + by = 1$, so the total is $60 + 12 = 72$.

8. The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab.

Solution:

For a 60° rotation,

$$37 = \frac{\sqrt{3}}{2}a + \frac{11}{2},$$

so $a = 21\sqrt{3}$. The first coordinate is then

$$b = \frac{a}{2} - \frac{11\sqrt{3}}{2} = 5\sqrt{3}.$$

The opposite orientation changes both signs and leaves the product unchanged. Hence $ab = (21\sqrt{3})(5\sqrt{3}) = 315$.

9. A solitaire game is played as follows. Six distinct pairs of matched tiles are placed in a bag. The player randomly draws tiles one at a time from the bag and retains them, except that matching tiles are put aside as soon as they appear in the player's hand. The game ends if the player ever holds three tiles, no two of which match; otherwise the drawing continues until the bag is empty. The probability that the bag will be emptied is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

Let $F(r,h)$ be the chance of success with r unseen pairs and $h \leq 2$ unmatched tiles held. Among $2r+h$ remaining tiles, h close an open pair and $2r$ open a new pair; the latter move fails when $h=2$. Thus

$$\begin{aligned} F(r,h) = &\frac{hF(r,h-1)}{2r+h} \\ &+ \frac{2rF(r-1,h+1)}{2r+h}, \end{aligned}$$

omitting the second term when $h=2$, with $F(0,h)=1$. Evaluating this three-state recursion gives

$$\begin{aligned} F(r,0): \quad &1,\ 1,\ \frac{3}{5},\ \frac{9}{35}, \\ &\frac{3}{35},\ \frac{9}{385} \end{aligned}$$

for $r=1,2,\ldots,6$. Hence $\frac{p}{q}=\frac{9}{385}$ and $p+q=394$.

10. In triangle ABC, angle C is a right angle and the altitude from C meets $\overline{AB}$ at D. The lengths of the sides of $\triangle ABC$ are integers, $BD = 29^3$, and $\cos B = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Write $\frac{BC}{AB} = \frac{m}{n}$ in lowest terms and $BC = km,\ AB = kn$. Similarity gives

$$29^3 = BD = \frac{BC^2}{AB} = \frac{km^2}{n}.$$

Hence $n \mid k$; writing $k = nt$ gives $tm^2 = 29^3$. The possibility $m = 1$ cannot be a leg of a nondegenerate integer right triangle, so $m = 29$. The primitive triple with leg 29 satisfies $(n-u)(n+u) = 29^2$, giving $n = 421$ and $u = 420$. Therefore $m + n = 29 + 421 = 450$.

11. Ninety-four bricks, each measuring $4'' \times 10'' \times 19''$, are to be stacked one on top of another to form a tower 94 bricks tall. Each brick can be oriented so it contributes $4''$, $10''$, or $19''$ to the total height of the tower. How many different tower heights can be achieved using all 94 of the bricks?

Solution:

If b bricks use height 10 and c use height 19, the total is $376 + 3(2b + 5c)$, where $b, c \geq 0$ and $b + c \leq 94$. For fixed c, the value $v = 2b + 5c$ runs by twos from $5c$ to $188 + 3c$. The intervals for c and $c + 2$ join through $c = 92$. Consequently every integer $0 \leq v \leq 470$ occurs except

$$1, 3, 463, 466, 468, 469.$$

There are $471 - 6 = 465$ distinct heights.

12. A fenced, rectangular field measures 24 meters by 52 meters. An agricultural researcher has 1994 meters of fence that can be used for internal fencing to partition the field into congruent, square test plots. The entire field must be partitioned, and the sides of the squares must be parallel to the edges of the field. What is the largest number of square test plots into which the field can be partitioned using all or some of the 1994 meters of fence?

Solution:

Let the grid have $6k$ rows and $13k$ columns, so each square has side $\frac{4}{k}$. The internal vertical and horizontal fences have total length

$$\begin{aligned} &52(6k-1)+24(13k-1) \\ &= 624k-76. \end{aligned}$$

Requiring this to be at most 1994 gives $k \leq 3$. With $k=3$, the number of plots is $(6k)(13k)=78k^2=702$.

13. The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

Solution:

Let $\zeta = \frac{x}{13x-1}$, so $\zeta^{10} = -1$ and

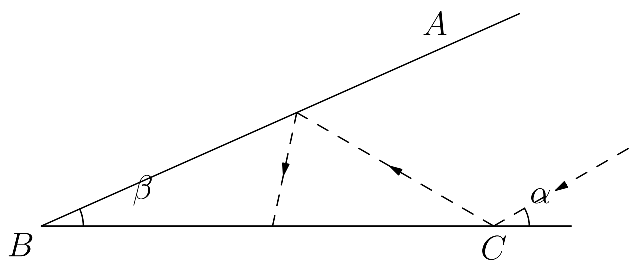
$$x = \frac{\zeta}{13\zeta - 1},$$
$$\frac{1}{x} = 13 - \zeta^{-1}.$$

Since $|\zeta| = 1$,

$$\frac{1}{|x|^2} = |13 - \zeta^{-1}|^2$$
$$= 170 - 13(\zeta + \overline{\zeta}).$$

Summing one value for each of the five conjugate pairs gives $5 \cdot 170$ minus 13 times the sum of all ten roots of $z^{10} + 1$, which is 0. The requested value is 850.

14. A beam of light strikes $\overline{BC}$ at point C with angle of incidence $\alpha = 19.94^{\circ}$ and reflects with an equal angle of reflection as shown. The light beam continues its path, reflecting off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule: angle of incidence equals angle of reflection. Given that $\beta = \frac{\alpha}{10} = 1.994^{\circ}$ and $AB = BC$, determine the number of times the light beam will bounce off the two line segments. Include the first reflection at C in your count.



Solution:

Unfold the path at each bounce, so the beam becomes one straight line crossing successive reflected copies of the angle at B. After k reflections beyond the initial reflection at C, the next boundary ray makes angle $k\beta$ with the original one. Using $AB = BC$, the crossing remains on the finite segments exactly while

$$k\beta \leq 180^{\circ} - 2\alpha.$$

Therefore

$$k \leq \frac{180 - 2(19.94)}{1.994} \approx 70.27,$$

so there are 70 further reflections. Including the first reflection at C gives 71.

15. Given a point P on a triangular piece of paper ABC, consider the creases that are formed in the paper when A, B, and C are folded onto P. Let us call P a fold point of $\triangle ABC$ if these creases, which number three unless P is one of the vertices, do not intersect. Suppose that $AB = 36, AC = 72$, and $\angle B = 90^\circ$. Then the area of the set of all fold points of $\triangle ABC$ can be written in the form $q\pi - r\sqrt{s}$, where q, r, and s are positive integers and s is not divisible by the square of any prime. What is $q + r + s$?

Solution:

The creases for two vertices meet at the circumcenter of the triangle formed with P. This intersection lies off the paper exactly when the angle at P is obtuse, so the fold-point locus is the intersection of the three diameter disks for AB, BC, CA. The CA disk contains the entire right triangle, leaving the intersection of the AB and BC disks.

Here $BC = \sqrt{72^2 - 36^2} = 36\sqrt{3}$. The two relevant radii are 18 and $18\sqrt{3}$, and their lens consists of circular segments with central angles 120° and 60°. Its area is

$$\begin{aligned}&\left(\frac{\pi}{3}\cdot 18^2 \right.\\ &\qquad\left. -\frac{1}{2}\cdot 18^2 \sin 120^\circ\right)\\ &+\left(\frac{\pi}{6}(18\sqrt{3})^2\right.\\ &\qquad\left.-\frac{1}{2}(18\sqrt{3})^2 \sin 60^\circ\right)\\ &= 270\pi - 324\sqrt{3}.\end{aligned}$$

Thus $q + r + s$ equals $270 + 324 + 3 = 597$.

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