1993 AIME Solutions

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1. How many even integers between 4000 and 7000 have four different digits?

Solution:

The thousands digit is $4, 5,$ or $6.$ If it is 4 or $6,$ the units digit has 4 choices among $0, 2, 4, 6, 8,$ after which the hundreds and tens digits have 8 and 7 choices. These two cases contribute $2 \cdot 4 \cdot 8 \cdot 7 = 448.$ If the thousands digit is $5,$ all 5 even units digits are available, contributing $5 \cdot 8 \cdot 7 = 280.$ Thus the total is $448 + 280 = 728.$

2. During a recent campaign for office, a candidate made a tour of a country which we assume lies in a plane. On the first day of the tour he went east, on the second day he went north, on the third day west, on the fourth day south, on the fifth day east, etc. If the candidate went $\frac{n^2}{2}$ miles on the nth day of this tour, how many miles was he from his starting point at the end of the 40th day?

Solution:

Index the ten cycles by $k = 0, 1, \ldots, 9$. The horizontal displacement is

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{9}\left((4k+1)^2 - (4k+3)^2\right)\\
&= \sum_{k=0}^{9}(-8k-4)\\
&= -400.
\end{aligned}$$

Similarly, the vertical displacement is

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{9}\left((4k+2)^2 - (4k+4)^2\right)\\
&= \sum_{k=0}^{9}(-8k-6)\\
&= -420.
\end{aligned}$$

Therefore the distance from the start is $\sqrt{400^2 + 420^2} = 580.$

3. The table below displays some of the results of last summer's Frostbite Falls Fishing Festival, showing how many contestants caught n fish for various values of n.

n	0	1	2	3	$\dots$
contestants	9	5	7	23	$\dots$

n	13	14	15
contestants	5	2	1

In the newspaper story covering the event, it was reported that

(a) the winner caught 15 fish;
(b) those who caught 3 or more fish averaged 6 fish each;
(c) those who caught 12 or fewer fish averaged 5 fish each.

What was the total number of fish caught during the festival?

Solution:

The $9+5+7=21$ contestants below 3 fish caught $5+14=19$ fish. Thus condition (b) gives $T-19=6(N-21)$, or $T=6N-107$. The $5+2+1=8$ contestants above 12 fish caught $65+28+15=108$ fish, so condition (c) gives $T-108=5(N-8)$, or $T=5N+68$. Hence $N=175$ and $T=943$.

4. How many ordered four-tuples of integers (a, b, c, d) with $0 < a < b < c < d < 500$ satisfy $a + d = b + c$ and $bc - ad = 93$?

Solution:

Let $x = b - a = d - c > 0$ and $y = c - a$. Then

$$\begin{aligned} bc - ad &= (a + x)c - a(c + x) \\ &= x(c - a) = xy = 93. \end{aligned}$$

Since $b < c$, we need $x < y$. The positive factor pairs are $(x, y) = (1, 93)$ and $(3, 31)$. For the first, $d = a + 94 < 500$ gives 405 choices for a. For the second, $d = a + 34 < 500$ gives 465 choices. The total is $405 + 465 = 870$.

5. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x - n)$. What is the coefficient of x in $P_{20}(x)$?

Solution:

The accumulated shift is $1 + 2 + \cdots + 20 = 210$, so $P_{20}(x) = P_0(x - 210)$. The coefficient of x is therefore

$$\begin{aligned} &3(210)^2 - 2(313)(210) - 77 \\ &= 132300 - 131460 - 77 \\ &= 763. \end{aligned}$$

6. What is the smallest positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers?

Solution:

The sums of 9 and 11 consecutive integers are divisible by 9 and 11, so the desired number is a multiple of 99. A sum of 10 consecutive integers has the form $10a + 45$, hence is congruent to $5 \pmod{10}$. The first multiple of 99 ending in 5 is $5 \cdot 99 = 495$, and each of the three required representations then exists.

7. Three numbers, a_1, a_2, a_3, are drawn randomly and without replacement from the set $\{1, 2, 3, \ldots, 1000\}$. Three other numbers, b_1, b_2, b_3, are then drawn randomly and without replacement from the remaining set of 997 numbers. Let p be the probability that, after a suitable rotation, a brick of dimensions $a_1 \times a_2 \times a_3$ can be enclosed in a box of dimensions $b_1 \times b_2 \times b_3$, with the sides of the brick parallel to the sides of the box. If p is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

Solution:

After the six distinct values are fixed and sorted, each of the $\binom{6}{3} = 20$ assignments of three values to the brick is equally likely. The sorted brick dimensions fit the sorted box dimensions exactly when, in every prefix of the resulting word of three a's and three b's, the number of a's is at least the number of b's. There are $C_3 = 5$ such words. Thus $p = \frac{5}{20} = \frac{1}{4}$, and the requested sum is $1 + 4 = 5$.

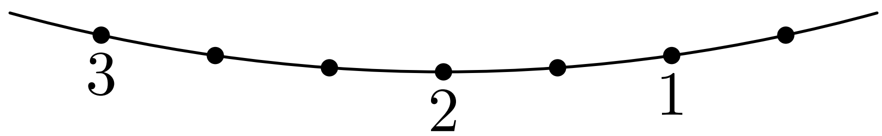
8. Let S be a set with six elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a, c\}, \{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}, \{a, c\}$.

Solution:

For an ordered pair (A, B) with $A \cup B = S$, each element has three possible memberships: A only, B only, or both. This gives $3^6 = 729$ ordered pairs. Swapping A and B fixes only the pair $A = B = S$. Therefore the number of unordered pairs is

$$\frac{729+1}{2} = 365.$$

9. Two thousand points are given on a circle. Label one of the points 1. From this point, count 2 points in the clockwise direction and label this point 2. From the point labeled 2, count 3 points in the clockwise direction and label this point 3. (See figure.) Continue this process until the labels $1, 2, 3, \ldots, 1993$ are all used. Some of the points on the circle will have more than one label and some points will not have a label. What is the smallest integer that labels the same point as 1993?



Solution:

Label j is displaced

$$2 + 3 + \cdots + j = \frac{j(j+1)}{2} - 1$$

points clockwise from label 1. Thus it shares the point of label 1993 exactly when $j(j+1) \equiv 1993 \cdot 1994$ modulo 4000, or equivalently when $j(j+1) \equiv 2042 \pmod{4000}$. Modulo 125, the solutions are $j \equiv 6, 118$, and modulo 32, they are $j \equiv 9, 22$. Combining these by the Chinese Remainder Theorem gives

$$j \equiv 118, 1993, \\ 2006, 3881 \pmod{4000}.$$

The smallest positive possibility is 118.

10. Euler's formula states that for a convex polyhedron with V vertices, E edges, and F faces, $V - E + F = 2$. A particular convex polyhedron has 32 faces, each of which is either a triangle or a pentagon. At each of its V vertices, T triangular faces and P pentagonal faces meet. What is the value of $100P + 10T + V$?

Solution:

Let x be the number of triangular faces, so there are $32 - x$ pentagons and

$$E = \frac{3x + 5(32 - x)}{2} = 80 - x.$$

Euler's formula gives $V + x = 50$. Counting face-vertex incidences yields

$$\begin{aligned} TV &= 3x = 150 - 3V, \\ PV &= 5(32 - x) = 5V - 90. \end{aligned}$$

Hence $V(T+3) = 150$ and $V(5-P) = 90$. Also $18 \le V \le 50$, so the only common divisor of 150 and 90 in that range is $V = 30$. Then $T = 2$ and $P = 2$, giving $100P + 10T + V = 250$.

11. Alfred and Bonnie play a game in which they take turns tossing a fair coin. The winner of a game is the first person to obtain a head. Alfred and Bonnie play this game several times with the stipulation that the loser of a game goes first in the next game. Suppose that Alfred goes first in the first game, and that the probability that he wins the sixth game is $\frac{m}{n}$, where m and n are relatively prime positive integers. What are the last three digits of $m+n$?

Solution:

Alfred wins a game with probability $\frac{2}{3}$ when he starts and $\frac{1}{3}$ when Bonnie starts. Since the loser starts the next game,

$$\begin{aligned} p_{r+1} &= \frac{2}{3}(1-p_r) + \frac{1}{3}p_r \\ &= \frac{2}{3} - \frac{1}{3}p_r, \qquad p_1 = \frac{2}{3}. \end{aligned}$$

Thus $p_r - \frac{1}{2} = \frac{(-\frac{1}{3})^{r-1}}{6}$. In particular, $p_6 = \frac{1}{2} - \frac{1}{1458} = \frac{364}{729}$. Therefore $m+n = 1093$, whose last three digits are 093.

12. The vertices of $\triangle ABC$ are $A = (0,0), B = (0,420)$, and $C = (560,0)$. The six faces of a die are labeled with two A's, two B's, and two C's. Point $P_1 = (k,m)$ is chosen in the interior of $\triangle ABC$, and points $P_2, P_3, P_4, \dots$ are generated by rolling the die repeatedly and applying the rule: If the die shows label L, where $L \in \{A,B,C\}$, and P_n is the most recently obtained point, then P_{n+1} is the midpoint of $\overline{P_nL}$. Given that $P_7 = (14,92)$, what is $k+m$?

Solution:

Let X and Y be the sums of the weights $1,2,4,8,16,32$ assigned to rolls of C and B, respectively. Iterating the midpoint rule gives

$$\begin{aligned} 64P_7 = P_1 + X(560,0) \\ + Y(0,420). \end{aligned}$$

Hence $k = 896 - 560X$. Because P_1 is interior, $0 < k < 560$, forcing $X = 1$ and $k = 336$. The triangle inequality for its coordinates then gives $0 < m < 168$. Since $m = 5888 - 420Y$, the only possible integer Y is 14, giving $m = 8$. Therefore $k + m = 344$.

13. Jenny and Kenny are walking in the same direction, Kenny at 3 feet per second and Jenny at 1 foot per second, on parallel paths that are 200 feet apart. A tall circular building 100 feet in diameter is centered midway between the paths. At the instant when the building first blocks the line of sight between Jenny and Kenny, they are 200 feet apart. Let t be the amount of time, in seconds, before Jenny and Kenny can see each other again. If t is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

Solution:

At first blockage the walkers are vertically aligned on the tangent $x = -50$. After t seconds their positions may be written as $(-50 + t, 100)$ and $(-50 + 3t, -100)$. The distance from the origin to their connecting line is

$$\frac{|10000 - 400t|}{\sqrt{4t^2 + 40000}}.$$

At the second tangency this equals 50. Squaring and simplifying gives

$$(100 - 4t)^2 = t^2 + 10000,$$

so $t(15t - 800) = 0$. The positive time is $t = \frac{160}{3}$, and the requested sum is $160 + 3 = 163.$

14. A rectangle that is inscribed in a larger rectangle (with one vertex on each side) is called unstuck if it is possible to rotate (however slightly) the smaller rectangle about its center within the confines of the larger. Of all the rectangles that can be inscribed unstuck in a 6 by 8 rectangle, the smallest perimeter has the form $\sqrt{N}$, for a positive integer N. Find N.

Solution:

By central symmetry, write consecutive vertices of an unstuck inscribed rectangle as $(4, y), (-x, 3), (-4, -y), (x, -3)$ with $x, y \geq 0$. Equal half-diagonals give $16 + y^2 = x^2 + 9$, so $x^2 - y^2 = 7$. If its side lengths are a, b, then

$$\begin{aligned} a^2 + b^2 &= 4(16 + y^2), \\ ab &= 24 + 2xy. \end{aligned}$$

Therefore its perimeter $Q = 2(a + b)$ satisfies

$$\begin{aligned} Q^2 &= 4(a^2 + b^2 + 2ab) \\ &= 448 + 16y(y + x) \geq 448. \end{aligned}$$

Equality occurs at $y = 0,\ x = \sqrt{7}$, which gives a non-axis-aligned, hence unstuck, rectangle. Thus the minimum perimeter is $\sqrt{448}$ and $N = 448$.

15. Let $\overline{CH}$ be an altitude of $\triangle ABC$. Let R and S be the points where the circles inscribed in the triangles ACH and BCH are tangent to $\overline{CH}$. If $AB = 1995$, $AC = 1994$, and $BC = 1993$, then RS can be expressed as $\frac{m}{n}$, where m and n are relatively prime integers. Find $m + n$.

Solution:

Let $h = CH$. In right triangle ACH, the tangent length from H to its incircle is $\frac{AH+h-AC}{2}$; in triangle BCH, it is $\frac{BH+h-BC}{2}$. Hence

$$RS = \frac{1}{2}\left|AH - BH - AC + BC\right|.$$

The projection formula gives

$$\begin{aligned} AH - BH &= \frac{AC^2 - BC^2}{AB} \\ &= \frac{1994^2 - 1993^2}{1995} \\ &= \frac{3987}{1995}. \end{aligned}$$

Since $AC - BC = 1$,

$$RS = \frac{1}{2}\left(\frac{3987}{1995} - 1\right) = \frac{332}{665}.$$

Thus $m + n = 332 + 665 = 997.$

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