1992 AIME Solutions

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1. Find the sum of all positive rational numbers that are less than 10 and that have denominator 30 when written in lowest terms.

Solution:

The numbers are $\frac{k}{30}$ for $1 \leq k < 300$ and $\gcd(k, 30) = 1$. There are $\varphi(30) = 8$ eligible residues in each block of 30, and their sum is $\frac{30\varphi(30)}{2} = 120$. Thus the sum of all eligible numerators is

$$\begin{aligned} \sum_{q=0}^{9}(8 \cdot 30q + 120) &= 8 \cdot 30 \cdot 45 \\ &\quad + 10 \cdot 120 \\ &= 12000. \end{aligned}$$

Dividing by 30 gives 400.

2. A positive integer is called ascending if, in its decimal representation, there are at least two digits and each digit is less than any digit to its right. How many ascending positive integers are there?

Solution:

The digit 0 cannot occur, because it would have to be the first digit and leading zeroes are not part of a decimal representation. Every subset of at least two digits from $\{1, \ldots, 9\}$ gives exactly one ascending integer when written in increasing order. Hence the number is

$$\begin{aligned} 2^9 - \binom{9}{0} - \binom{9}{1} &= 512 - 1 - 9 \\ &= 502. \end{aligned}$$

3. A tennis player computes her win ratio by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly 0.500. During the weekend, she plays four matches, winning three and losing one. At the end of the weekend, her win ratio is greater than 0.503. What's the largest number of matches she could've won before the weekend began?

Solution:

If she initially had w wins, then she had played $2w$ matches. The final condition is

$$\frac{w+3}{2w+4} > \frac{503}{1000}.$$

Cross-multiplication gives $1000w + 3000 > 1006w + 2012$, so $6w < 988$ and $w < 164\frac{2}{3}$. The largest possible integer is 164.

4. In Pascal's Triangle, each entry is the sum of the two entries above it. The first few rows of the triangle are shown below.

$$
\begin{array}{c|ccccccc}
0 & 1 \\
1 & 1 & 1 \\
2 & 1 & 2 & 1 \\
3 & 1 & 3 & 3 & 1 \\
4 & 1 & 4 & 6 & 4 & 1 \\
5 & 1 & 5 & 10 & 10 & 5 & 1 \\
6 & 1 & 6 & 15 & 20 & 15 & 6 & 1
\end{array}
$$

In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3 : 4 : 5$?

Solution:

For three consecutive entries beginning at position k,

$$
\begin{aligned}
\frac{\binom{n}{k+1}}{\binom{n}{k}} &= \frac{n-k}{k+1} = \frac{4}{3}, \\
\frac{\binom{n}{k+2}}{\binom{n}{k+1}} &= \frac{n-k-1}{k+2} = \frac{5}{4}.
\end{aligned}
$$

Thus $3n = 7k + 4$ and $4n = 9k + 14$. Solving gives $k = 26$ and $n = 62$.

5. Let S be the set of all rational numbers $r, 0 < r < 1$, that have a repeating decimal expansion in the form

$$0.abcabcabc\ldots = 0.\overline{abc},$$

where the digits a, b, and c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

Solution:

Every element is $\frac{N}{999}$ for $1 \leq N \leq 998$. Any a coprime to 999 occurs as a reduced numerator with denominator 999, giving $\varphi(999) = 648$ values. If a is divisible by 3 but not 37, it can be coprime to a reduced denominator only when that denominator is 37; this adds the 12 multiples of 3 below 37. A numerator divisible by 37 would need a denominator dividing 27 and larger than it, so no further values occur. Therefore the total is $648 + 12 = 660$.

6. For how many pairs of consecutive integers in $\{1000, 1001, 1002, \ldots, 2000\}$ is no carrying required when the two integers are added?

Solution:

Write the smaller number as $1abc$. If $c \neq 9$, then $c \leq 4$ and the unchanged digits a, b are each at most 4, giving $5^3 = 125$ pairs. If $c = 9$ but $b \neq 9$, then $a, b \leq 4$, giving 25 pairs. If $b = c = 9$ but $a \neq 9$, there are 5 choices for a. Finally, $1999 + 2000$ also needs no carry. The total is $125 + 25 + 5 + 1 = 156$.

7. Faces ABC and BCD of tetrahedron $ABCD$ meet at an angle of 30°. The area of face ABC is 120, the area of face BCD is 80, and $BC = 10$. Find the volume of the tetrahedron.

Solution:

The altitudes to BC in faces ABC and BCD are $\frac{2(120)}{10} = 24$ and $\frac{2(80)}{10} = 16$, respectively. Because the dihedral angle is 30°, the perpendicular height from D to plane ABC is $16 \sin 30^\circ = 8$. Using face ABC as the base, the volume is

$$\frac{1}{3}(120)(8) = 320.$$

8. For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots)$, whose nth term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{92} = 0$. Find a_1.

Solution:

A quadratic sequence with second difference 1 has leading coefficient $\frac{1}{2}$. Since its values vanish at indices 19 and 92,

$$a_n = \frac{1}{2}(n - 19)(n - 92).$$

Therefore $a_1 = \frac{1}{2}(-18)(-91) = 819$.

9. Trapezoid $ABCD$ has sides $AB = 92, BC = 50, CD = 19$, and $AD = 70$, with AB parallel to CD. A circle with center P on AB is drawn tangent to BC and AD. Given that $AP = \frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Solution:

Put $A = (0, 0), B = (92, 0)$, and let the height of the trapezoid be h. If $P = (p, 0)$, its perpendicular distances to legs AD and BC are $\frac{hp}{70}$ and $\frac{h(92-p)}{50}$, respectively. Tangency to both legs makes these equal, so

$$\frac{p}{70} = \frac{92 - p}{50}.$$

Hence $120p = 6440$ and $AP = p = \frac{161}{3}$. Therefore $m + n = 161 + 3 = 164.$

10. Consider the region A in the complex plane that consists of all points z such that both $\frac{z}{40}$ and $\frac{40}{\overline{z}}$ have real and imaginary parts between 0 and 1, inclusive. What is the integer that is nearest the area of A?

Solution:

Write $z = x + iy$. The condition on $\frac{z}{40}$ gives $0 \le x, y \le 40$. Since

$$\frac{40}{\overline{z}} = \frac{40x}{x^2 + y^2} + i\frac{40y}{x^2 + y^2},$$

the other condition requires $x^2 + y^2 \ge 40x$ and $x^2 + y^2 \ge 40y$. Thus, within the square, we remove two semicircles of radius 20.

Their overlap is the lens formed by two radius-20 circles whose centers are $20\sqrt{2}$ apart. Its area is $200\pi - 400$. Hence the removed union has area $400\pi - (200\pi - 400)$, or $200\pi + 400$. Therefore

$$\begin{aligned} \text{area}(A) &= 1600 - (200\pi + 400) \\ &= 1200 - 200\pi \\ &\approx 571.68. \end{aligned}$$

The nearest integer is 572.

11. Lines l_1 and l_2 both pass through the origin and make first-quadrant angles of $\frac{\pi}{70}$ and $\frac{\pi}{54}$ radians, respectively, with the positive x-axis. For any line l, the transformation $R(l)$ produces another line as follows: l is reflected in l_1, and the resulting line is reflected in l_2. Let $R^{(1)}(l) = R(l)$ and $R^{(n)}(l) = R(R^{(n-1)}(l))$. Given that l is the line $y = (\frac{19}{92})x$, find the smallest positive integer m for which $R^{(m)}(l) = l$.

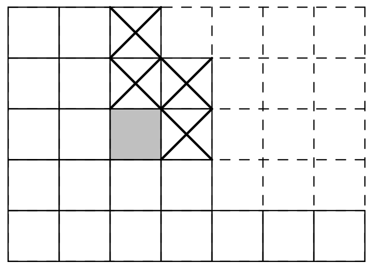
Solution:

The composition is rotation through

$$2\left(\frac{\pi}{54} - \frac{\pi}{70}\right) = \frac{8\pi}{945}.$$

A line through the origin is unchanged by a rotation exactly when the rotation angle is a multiple of π. Thus m must satisfy $\frac{8m}{945} \in \mathbb{Z}$. Since $\gcd(8, 945) = 1$, the least such m is 945.

12. In a game of Chomp, two players alternately take bites from a 5-by-7 grid of unit squares. To take a bite, a player chooses one of the remaining squares, then removes ("eats") all squares in the quadrant defined by the left edge (extended upward) and the lower edge (extended rightward) of the chosen square. For example, the bite determined by the shaded square in the diagram would remove the shaded square and the four squares marked by $\times$. (The squares with two or more dotted edges have been removed from the original board in previous moves.)



The object of the game is to make one's opponent take the last bite. The diagram shows one of the many subsets of the set of 35 unit squares that can occur during the game of Chomp. How many different subsets are there in all? Include the full board and empty board in your count.

Solution:

After any sequence of bites, the remaining squares form a lower-left order ideal: the seven column heights are nonincreasing integers between 0 and 5. Conversely, every such boundary can be produced and corresponds to a lattice path across a 5-by-7 rectangle. Each path consists of 5 vertical and 7 horizontal steps, so the number of states, including full and empty, is

$$\binom{12}{5} = 792.$$

13. Triangle ABC has $AB = 9$ and $BC : AC = 40 : 41$. What's the largest area that this triangle can have?

Solution:

Set $BC = 40t,\ AC = 41t$, and $x = \cos C$. The Law of Cosines gives

$$81 = t^2(3281 - 3280x),$$

while the area is $820t^2\sqrt{1 - x^2}$. Hence it equals

$$820 \cdot 81\, \frac{\sqrt{1 - x^2}}{3281 - 3280x}.$$

Differentiating its logarithm shows the maximum occurs at $x = \frac{3280}{3281}$. Then $\sqrt{1 - x^2} = \frac{81}{3281}$ and $3281 - 3280x = \frac{6561}{3281}$, so the maximum area is 820.

14. In triangle ABC, A', B', and C' are on the sides BC, AC, and AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that

$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92,$$

find

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}.$$

Solution:

Let $\alpha + \beta + \gamma = 1$ be the barycentric coordinates of O. Then

$$x = \frac{AO}{OA'} = \frac{\beta + \gamma}{\alpha},$$
$$y = \frac{BO}{OB'} = \frac{\gamma + \alpha}{\beta},$$
$$z = \frac{CO}{OC'} = \frac{\alpha + \beta}{\gamma}.$$

Expanding both sides using $\alpha + \beta + \gamma = 1$ gives the standard identity

$$xyz = x + y + z + 2.$$

Since $x + y + z = 92$, the requested product is 94.

15. Define a positive integer n to be a factorial tail if there is some positive integer m such that the decimal representation of $m!$ ends with exactly n zeroes. How many positive integers less than 1992 are not factorial tails?

Solution:

The number of trailing zeroes is $f(m) = \sum_{j\geq 1} \lfloor \frac{m}{5^j} \rfloor$. Its positive distinct values occur at the multiples $5k$, and

$$f(5k) = k + f(k)$$

is strictly increasing with k. Now

$$\begin{aligned} f(1595) = 319 + 63 \\ + 12 + 2 = 396, \end{aligned}$$

so $f(7975) = 1595 + 396 = 1991$. For $k = 1596$, the same calculation gives $f(k) = 396$, hence $f(5k) = 1992$. Therefore exactly 1595 positive values through 1991 are factorial tails. Of the 1991 positive integers below 1992, the number omitted is $1991 - 1595 = 396$.

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