1991 AIME Solutions

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1. Find x^2+y^2 if x and y are positive integers such that

$$\begin{aligned} xy+x+y &= 71, \\ x^2y+xy^2 &= 880. \end{aligned}$$

Solution:

Let $s=x+y$ and $p=xy$. The equations become $s+p=71$ and $sp=880$, so s and p are the roots of

$$\begin{aligned} t^2-71t+880 &= 0, \\ (t-16)(t-55) &= 0. \end{aligned}$$

Because x and y are positive integers, $s=16$ and $p=55$; indeed, $x,y=5,11$. Therefore

$$\begin{aligned} x^2+y^2 &= s^2-2p \\ &= 16^2-2(55)=146. \end{aligned}$$

2. Rectangle $ABCD$ has sides $\overline{AB}$ of length 4 and $\overline{CB}$ of length 3. Divide $\overline{AB}$ into 168 congruent segments with points $A = P_0, P_1, \ldots, P_{168} = B$, and divide $\overline{CB}$ into 168 congruent segments with points $C = Q_0, Q_1, \ldots, Q_{168} = B$. For $1 \le k \le 167$, draw the segments $\overline{P_kQ_k}$. Repeat this construction on the sides $\overline{AD}$ and $\overline{CD}$, and then draw the diagonal $\overline{AC}$. Find the sum of the lengths of the 335 parallel segments drawn.

Solution:

Put $A = (0,0)$, $B = (4,0)$, and $C = (4,3)$. Then

$$P_k = \left(\frac{4k}{168}, 0\right),$$
$$Q_k = \left(4, 3 - \frac{3k}{168}\right),$$

so the 3-4-5 ratio gives

$$P_kQ_k = 5\left(1 - \frac{k}{168}\right).$$

Hence one construction has total length

$$5\sum_{k=1}^{167}\left(1 - \frac{k}{168}\right) = \frac{5 \cdot 167}{2}.$$

The construction on the other two sides has the same total, and $AC = 5$. Thus the requested sum is $5(167) + 5 = 840$.

3. Expanding $(1+0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\begin{aligned}&\binom{1000}{0}(0.2)^0+\binom{1000}{1}(0.2)^1\\&+\binom{1000}{2}(0.2)^2+\cdots\\&+\binom{1000}{1000}(0.2)^{1000}\\&=A_0+A_1+\cdots+A_{1000},\end{aligned}$$

where $A_k=\binom{1000}{k}(0.2)^k$ for $k=0,1,2,\ldots,1000$. For which k is A_k the largest?

Solution:

Consecutive terms satisfy

$$\frac{A_{k+1}}{A_k}=\frac{1000-k}{k+1}\cdot\frac{1}{5}.$$

This ratio exceeds 1 exactly when $1000-k>5k+5$, or $k<\frac{995}{6}$. Thus the terms increase through A_{166} and decrease afterward. Therefore the largest term is A_{166}.

4. How many real numbers x satisfy the equation

$$\frac{1}{5}\log_2 x = \sin(5\pi x)?$$

Solution:

Let $h(x) = \sin(5\pi x) - \frac{1}{5}\log_2 x$. Any root lies in $[2^{-5}, 2^5] = [\frac{1}{32}, 32]$.

For $x < 1$, a root can occur only on a negative half-wave of the sine. There are two such half-waves, $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$. On each, h is positive at both endpoints and negative at the midpoint, so there are two roots. Uniqueness on the descending half follows from monotonicity; on the ascending half,

$$\begin{aligned} h'' &= -25\pi^2 \sin(5\pi x) \\ &\quad + \frac{1}{5x^2 \ln 2} > 0, \end{aligned}$$

so h' increases and there is only one crossing. Thus there are 4 roots below 1.

Also $x = 1$ is a root. For $1 < x < 32$, roots can occur only on positive half-waves $(\frac{2m}{5}, \frac{2m+1}{5})$ with $m = 3, 4, \ldots, 79$. There are 77 of these. The value of h is negative at each endpoint and positive at the midpoint. The right half is strictly decreasing. On the left half,

$$\begin{aligned} h''' &= -125\pi^3 \cos(5\pi x) \\ &\quad - \frac{2}{5x^3 \ln 2} < 0, \end{aligned}$$

so h' is strictly concave and changes sign only once. Hence every such half-wave contributes exactly two roots. Therefore the total number is

$$4 + 1 + 2(77) = 159.$$

5. Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will $20!$ be the resulting product?

Solution:

The distinct primes dividing $20!$ are $2, 3, 5, 7, 11, 13, 17,$ and 19. If $\frac{a}{b}$ is in lowest terms and $ab = 20!$, the entire power of each of these eight primes must be assigned to either a or b. Thus there are 2^8 ordered coprime factorizations $ab = 20!$. Since $a \neq b$, exactly half have $0 < \frac{a}{b} < 1$. The number sought is $2^7 = 128$.

6. Suppose r is a real number for which

$$\left\lfloor r+\frac{19}{100}\right\rfloor+\left\lfloor r+\frac{20}{100}\right\rfloor+\left\lfloor r+\frac{21}{100}\right\rfloor+\cdots+\left\lfloor r+\frac{91}{100}\right\rfloor=546.$$

Find $\lfloor 100r\rfloor$. (For real x, $\lfloor x\rfloor$ is the greatest integer less than or equal to x.)

Solution:

Write $r=n+f$, where n is an integer and $0\le f<1$. There are 73 summands. Since $546=73\cdot 7+35$, we must have $n=7$, and exactly 35 of the numbers $f+\frac{19}{100},\dots,f+\frac{91}{100}$ have floor 1. These must be the terms with numerators $57,58,\dots,91$. Hence

$$f+\frac{56}{100}<1\le f+\frac{57}{100},$$

so $43\le 100f<44$. Therefore $\lfloor 100r\rfloor=700+43=743.$

7. Find A^2, where A is the sum of the absolute values of all roots of the following equation:

$$x=\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{x}}}}}.$$

Solution:

Let $f(t)=\sqrt{19}+\frac{91}{t}$, and let $\alpha>0>\beta$ be its fixed points. They satisfy

$$t^2-\sqrt{19}\,t-91=0.$$

A direct subtraction using $91=\alpha(\alpha-\sqrt{19})=\beta(\beta-\sqrt{19})$ gives

$$\frac{f(t)-\alpha}{f(t)-\beta}=\frac{\beta}{\alpha}\,\frac{t-\alpha}{t-\beta}.$$

The given equation is $f^5(x)=x$. If x were neither α nor β, iterating the displayed ratio five times would force $(\frac{\beta}{\alpha})^5=1$, which is impossible because $\frac{\beta}{\alpha}<0$. Thus the only roots are α and β.

Their absolute values sum to $\alpha-\beta$, the difference of the roots of the quadratic. Hence

$$A=\sqrt{(\sqrt{19})^2+4(91)}=\sqrt{383},$$

so $A^2=383$.

8. For how many real numbers a does the quadratic equation $x^2 + ax + 6a = 0$ have only integer roots for x?

Solution:

Let the integer roots be r, s. Vieta's formulas give $r + s = -a$ and $rs = 6a$, so $rs = -6(r + s)$. Therefore

$$(r + 6)(s + 6) = 36.$$

Conversely, every ordered integer factorization $uv = 36$ gives integer roots $r = u - 6,\ s = v - 6$ and $a = 12 - u - v$. Unordered positive factor pairs of 36 have sums $37, 20, 15, 13, 12$, and the corresponding negative factor pairs have their negatives as sums. These ten sums are distinct, so they give 10 distinct values of a.

9. Suppose that $\sec x + \tan x = \frac{22}{7}$ and that $\csc x + \cot x = \frac{m}{n}$, where $\frac{m}{n}$ is in lowest terms. Find $m + n$.

Solution:

Let $u = \sec x + \tan x = \frac{22}{7}$. Since $\sec x + \tan x$ and $\sec x - \tan x$ have product 1,

$$\begin{aligned} \sec x &= \frac{u + u^{-1}}{2}, \\ \tan x &= \frac{u - u^{-1}}{2}. \end{aligned}$$

Also

$$\begin{aligned} \csc x + \cot x &= \frac{1 + \cos x}{\sin x} \\ &= \frac{\sec x + 1}{\tan x} \\ &= \frac{u + 1}{u - 1}. \end{aligned}$$

Substituting $u = \frac{22}{7}$ gives $\frac{29}{15}$. Thus $m + n = 29 + 15 = 44$.

10. Two three-letter strings, aaa and bbb, are transmitted electronically. Each string is sent letter by letter. Due to faulty equipment, each of the six letters has a $\frac{1}{3}$ chance of being received incorrectly, as an a when it should have been a b, or as a b when it should be an a. However, whether a given letter is received correctly or incorrectly is independent of the reception of any other letter.

Let S_a be the three-letter string received when aaa is transmitted and let S_b be the three-letter string received when bbb is transmitted. Let p be the probability that S_a comes before S_b in alphabetical order. When p is written as a fraction in lowest terms, what is its numerator?

Solution:

At any position, the received letters agree with probability

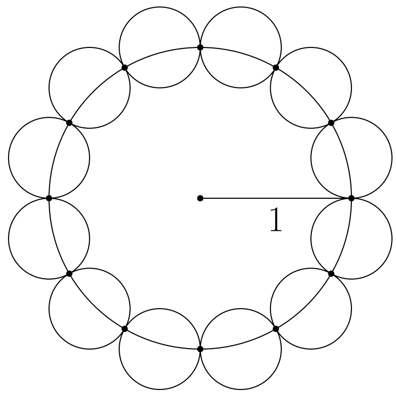
$$2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{4}{9}.$$

The favorable first difference, S_a receiving a and S_b receiving b, has probability $(\frac{2}{3})^2 = \frac{4}{9}$. It can occur in the first, second, or third position after zero, one, or two agreements. Hence

$$\begin{aligned} p &= \frac{4}{9}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2\right) \\ &= \frac{532}{729}. \end{aligned}$$

This fraction is in lowest terms, so its numerator is 532.

11. Twelve congruent disks are placed on a circle C of radius 1 in such a way that the twelve disks cover C, no two of the disks overlap, and so that each of the twelve disks is tangent to its two neighbors. The resulting arrangement of disks is shown in the figure below. The sum of the areas of the twelve disks can be written in the form $\pi(a - b\sqrt{c})$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.



Solution:

Let O be the center of C, let U, V be the centers of two neighboring disks, and let T be their tangency point. By the 12-fold symmetry, $\angle UOV = 30^\circ$, and OT bisects that angle. Also T is the midpoint of $\overline{UV}$, so triangle OUT is right at T. Since T lies on $C, OT = 1,$ while UT is the disk radius r. Thus

$$r = \tan 15^\circ = 2 - \sqrt{3}.$$

The total area is

$$\begin{aligned} 12\pi r^2 &= 12\pi(7 - 4\sqrt{3}) \\ &= \pi(84 - 48\sqrt{3}). \end{aligned}$$

Therefore $a + b + c = 84 + 48 + 3 = 135.$

12. Rhombus $PQRS$ is inscribed in rectangle $ABCD$ so that vertices $P, Q, R,$ and S are interior points on sides $\overline{AB}, \overline{BC}, \overline{CD},$ and $\overline{DA},$ respectively. It is given that $PB = 15,$ $BQ = 20, PR = 30,$ and $QS = 40.$ Let $\frac{m}{n}$, in lowest terms, denote the perimeter of $ABCD.$ Find $m + n.$

Solution:

Let the rectangle have width w and height $h,$ with $A = (0, 0)$ and $B = (w, 0).$ Then $P = (w - 15, 0)$ and $Q = (w, 20).$ The diagonals of the rhombus bisect each other at the rectangle's center $O = (\frac{w}{2}, \frac{h}{2}).$ Put $p = \overrightarrow{OP} = (\frac{w}{2} - 15, -\frac{h}{2}).$ Since $OP = 15,\ OQ = 20,$ and the rhombus diagonals are perpendicular, while $\overrightarrow{PQ} = (15, 20),$ we have

$$|p| = 15,$$
$$p \cdot (15, 20) = -225.$$

Solving these two equations, with the second coordinate negative because $h > 0,$ gives $p = (\frac{21}{5}, -\frac{72}{5}).$ Hence

$$w = 2\left(15 + \frac{21}{5}\right) = \frac{192}{5},$$
$$h = \frac{144}{5}.$$

The perimeter is $2(w + h) = \frac{672}{5},$ so $m + n = 672 + 5 = 677.$

13. A drawer contains a mixture of red socks and blue socks, at most 1991 in all. It so happens that, when two socks are selected randomly without replacement, there is a probability of exactly $\frac{1}{2}$ that both are red or both are blue. What is the largest possible number of red socks in the drawer that is consistent with this data?

Solution:

Let r, b be the two color counts and $n = r + b$. The probability of drawing different colors is also $\frac{1}{2}$, so

$$\frac{2rb}{\binom{n}{2}} = \frac{1}{2},$$
$$4rb = n(n-1).$$

Since $4rb = (r+b)^2 - (r-b)^2$, this becomes $(r-b)^2 = n$. Thus $n = k^2$ and, choosing red as the more numerous color,

$$r = \frac{k^2 + k}{2}.$$

The largest square at most 1991 is $44^2 = 1936$, giving $r = \frac{1936+44}{2} = 990.$

14. A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

Solution:

Let $2u$ be the central angle subtended by each 81-side, and put $t = 2\cos u$. The remaining arc has half-angle $\pi - 5u$, so the chord ratio gives

$$\begin{aligned}\frac{31}{81} &= \frac{\sin 5u}{\sin u} \\ &= t^4 - 3t^2 + 1.\end{aligned}$$

This yields $t^2 = \frac{25}{9}$ or $\frac{2}{9}$. Because $5u < \pi$, we have $t > 2\cos 36^\circ$, so $t = \frac{5}{3}$.

The three diagonals from A subtend the same minor angles as $2u, 3u,$ and $4u$. Relative to an 81-side, their length ratios are

$$\begin{aligned}\frac{\sin 2u}{\sin u} &= t, \\ \frac{\sin 3u}{\sin u} &= t^2 - 1, \\ \frac{\sin 4u}{\sin u} &= t^3 - 2t.\end{aligned}$$

Their sum is therefore

$$\begin{aligned}81(t^3 + t^2 - t - 1) &= 81\left(\frac{128}{27}\right) \\ &= 384.\end{aligned}$$

15. For positive integer n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers whose sum is 17. There is a unique positive integer n for which S_n is also an integer. Find this n.

Solution:

The two component sums are $\sum_{k=1}^{n}(2k-1) = n^2$ and $\sum_{k=1}^{n} a_k = 17$. Therefore the triangle inequality for vectors gives

$$\begin{aligned}\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2} \\ &\geq \sqrt{(n^2)^2 + 17^2} \\ &= \sqrt{n^4 + 289}.\end{aligned}$$

Equality is attainable by taking a_k proportional to $2k-1$, so $S_n = \sqrt{n^4 + 289}$.

If this is the integer m, then

$$(m - n^2)(m + n^2) = 289 = 17^2.$$

The factor pair $17, 17$ gives $n = 0$, while the pair $1, 289$ gives $m = 145$ and $n^2 = 144$. Thus the unique positive n is 12.

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