1990 AIME Solutions

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1. The increasing sequence $2, 3, 5, 6, 7, 10, 11, \ldots$ consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term of this sequence.

Solution:

Through $529,$ there are $\lfloor \sqrt{529} \rfloor = 23$ squares, 8 cubes, and 2 sixth powers counted in both groups. Thus the number of allowed terms at most 529 is

$$529 - 23 - 8 + 2 = 500.$$

But $529 = 23^2$ is excluded, while 528 is neither a square nor a cube. Therefore the 500 th term is $528.$

2. Find the value of

$$(52 + 6\sqrt{43})^{\frac{3}{2}} - (52 - 6\sqrt{43})^{\frac{3}{2}}.$$

Solution:

Since

$$52 \pm 6\sqrt{43} = (\sqrt{43} \pm 3)^2$$

and $\sqrt{43} > 3$, the expression is $(\sqrt{43}+3)^3 - (\sqrt{43}-3)^3$. Using $(x+y)^3 - (x-y)^3 = 6x^2y + 2y^3$ with $x = \sqrt{43}$ and $y = 3$ gives

$$\begin{aligned} 6(43)(3) + 2(27) &= 774 + 54 \\ &= 828. \end{aligned}$$

3. Let P_1 be a regular r-gon and P_2 be a regular s-gon $(r \geq s \geq 3)$ such that each interior angle of P_1 is $\frac{59}{58}$ as large as each interior angle of P_2. What is the largest possible value of s?

Solution:

The angle condition gives

$$\frac{\frac{r-2}{r}}{\frac{s-2}{s}} = \frac{59}{58},$$

which simplifies to $r(118 - s) = 116s$. Put $t = 118 - s$. Then

$$r = \frac{13688}{t} - 116.$$

In particular, t is a positive divisor of 13688, and maximizing $s = 118 - t$ means taking the least possible t. The choice $t = 1$ gives $s = 117$ and the positive integer $r = 13572$, which also satisfies $r \geq s$. Hence the largest possible s is 117.

4. Find the positive solution to

$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0.$$

Solution:

Set $y = x^2 - 10x$. Combining the first two fractions and clearing the nonzero denominators gives an equality between $(y - 37)(y - 69)$ and $(y - 29)(y - 45)$. Expanding and canceling y^2 yields $y = 39$. Thus $x^2 - 10x - 39 = 0$, so $(x - 13)(x + 3) = 0$. The positive solution is 13.

5. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $\frac{n}{75}$.

Solution:

The multiplicative partitions of 75 give exponent patterns $(74),\ (24,2),\ (14,4)$, and $(4,4,2)$. A number divisible by $75=3\cdot 5^2$ needs both primes 3 and 5, so the one-prime pattern is impossible. The two-prime patterns force very large powers of 3 and 5. The smallest three-prime candidate is

$$n=2^4\cdot 3^4\cdot 5^2.$$

It has $(4+1)(4+1)(2+1)=75$ divisors and is smaller than either two-prime candidate. Therefore

$$\frac{n}{75}=\frac{2^4\cdot 3^4\cdot 5^2}{3\cdot 5^2}=2^4\cdot 3^3=432.$$

6. A biologist wants to calculate the number of fish in a lake. On May 1 she catches a random sample of 60 fish, tags them, and releases them. On September 1 she catches a random sample of 70 fish and finds that 3 of them are tagged. To calculate the number of fish in the lake on May $1,$ she assumes that 25% of these fish are no longer in the lake on September 1 (because of death and emigrations), that 40% of the fish were not in the lake May 1 (because of births and immigrations), and that the number of untagged fish and tagged fish in the September 1 sample are representative of the total population. What does the biologist calculate for the number of fish in the lake on May 1?

Solution:

Of the 60 tagged fish, 75% remain in September, so 45 tagged fish remain. The sample estimates that tagged fish form $\frac{3}{70}$ of the September population, making that population

$$45\left(\frac{70}{3}\right) = 1050.$$

Of those fish, 60% were present in May, so 630 surviving May fish remain. These are 75% of the May population. Thus the estimated May population is $\frac{630}{0.75} = 840.$

7. A triangle has vertices $P = (-8, 5), Q = (-15, -19)$, and $R = (1, -7)$. The equation of the bisector of $\angle P$ can be written in the form $ax + 2y + c = 0$. Find $a + c$.

Solution:

We have $\overrightarrow{PQ} = (-7, -24)$ with length 25, and $\overrightarrow{PR} = (9, -12)$ with length 15. The sum of their unit vectors is

$$\begin{aligned} v = & \left(-\frac{7}{25}, -\frac{24}{25}\right) \\ & + \left(\frac{3}{5}, -\frac{4}{5}\right) \\ = & \frac{1}{25}(8, -44), \end{aligned}$$

so the angle bisector has direction $(2, -11)$. A normal vector is $(11, 2)$. Through $P = (-8, 5)$, its equation is

$$11(x + 8) + 2(y - 5) = 0,$$

or $11x + 2y + 78 = 0$. Hence $a + c = 11 + 78 = 89$.

8. In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules:

(1) The marksman first chooses a column from which a target is to be broken.

(2) The marksman must then break the lowest remaining target in the chosen column.

If the rules are followed, in how many different orders can the eight targets be broken?

Solution:

Once the chosen column is known at each shot, the target within that column is forced. Thus every valid order corresponds to an arrangement of three symbols from the first column, three from the second, and two from the third. The number of such arrangements is

$$\frac{8!}{3!\,3!\,2!} = 560.$$

9. A fair coin is to be tossed 10 times. Let $\frac{i}{j}$, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i + j$.

Solution:

Let u_n be the number of length-n toss strings with no consecutive heads. A valid string ending in tails is obtained by appending T to any valid length-$(n-1)$ string, while one ending in heads is obtained by appending TH to any valid length-$(n-2)$ string. Hence $u_n = u_{n-1} + u_{n-2}$, with $u_1 = 2$ and $u_2 = 3$. This gives $u_{10} = 144$. The probability is $\frac{144}{2^{10}} = \frac{9}{64}$, so $i + j = 9 + 64 = 73$.

10. The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C?

Solution:

The arguments of products in C are

$$2\pi\left(\frac{a}{18} + \frac{b}{48}\right) = \frac{2\pi(8a + 3b)}{144}.$$

Since $\gcd(8, 3) = 1$, the residues $8a + 3b$ generate every residue modulo 144. Thus C is precisely the set of 144th roots of unity. Equivalently, its order is $\operatorname{lcm}(18, 48) = 144$.

11. Someone observed that $6! = 8 \cdot 9 \cdot 10$. Find the largest positive integer n for which $n!$ can be expressed as the product of $n-3$ consecutive positive integers.

Solution:

The $n-3$ consecutive integers beginning at 4 have product

$$4 \cdot 5 \cdots n = \frac{n!}{6},$$

while those beginning at 5 have product

$$\begin{aligned} 5 \cdot 6 \cdots (n+1) &= \frac{(n+1)!}{24} \\ &= \frac{n+1}{24} n!. \end{aligned}$$

For $n = 23$, the latter product equals $n!$, so 23 works. For every $n \geq 24$, the product beginning at 4 is below $n!$, the product beginning at 5 is above $n!$, and the product strictly increases with its initial term. Hence no $n \geq 24$ works, and the largest possible value is 23.

12. A regular 12-gon is inscribed in a circle of radius 12. The sum of the lengths of all sides and diagonals of the 12-gon can be written in the form

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6},$$

where $a, b, c,$ and d are positive integers. Find $a + b + c + d$.

Solution:

For step sizes $k = 1, \ldots, 5$, there are 12 chords of length $24\sin(\frac{k\pi}{12})$, and there are 6 diameters of length 24. The five chord lengths are

$$\begin{gathered} 6(\sqrt{6} - \sqrt{2}), \quad 12, \quad 12\sqrt{2}, \\ 12\sqrt{3}, \quad 6(\sqrt{6} + \sqrt{2}). \end{gathered}$$

Their sum U is

$$\begin{aligned} U = {} & 12 + 12\sqrt{2} \\ & + 12\sqrt{3} + 12\sqrt{6}. \end{aligned}$$

Therefore the total is

$$\begin{aligned} 12U + 6(24) = {} & 288 + 144\sqrt{2} \\ & + 144\sqrt{3} \\ & + 144\sqrt{6}. \end{aligned}$$

Hence $a = 288$ and $b = c = d = 144$, so $a + b + c + d = 720$.

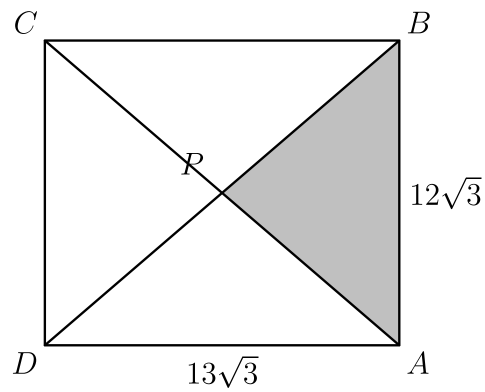
13. Let T be the set of powers 9^k, where k is an integer with $0 \le k \le 4000.$ Given that 9^{4000} has 3817 digits and that its first (leftmost) digit is $9,$ how many elements of T have 9 as their leftmost digit?

Solution:

For $k \geq 1,$ the number 9^k begins with 9 exactly when it has the same number of digits as $9^{k-1}.$ Indeed, multiplying a d-digit number by 9 without creating a new digit produces a d-digit number at least $9 \cdot 10^{d-1}.$

Starting from the one-digit number $9^0,$ the digit count reaches 3817 after 4000 multiplications. Thus it increases on 3816 steps and stays unchanged on $4000 - 3816 = 184$ steps. Since $9^0 = 1$ does not begin with $9,$ exactly 184 elements of T do.

14. The rectangle $ABCD$ below has dimensions $AB = 12\sqrt{3}$ and $BC = 13\sqrt{3}$. Diagonals AC and BD intersect at P. If triangle ABP is cut out and removed, edges AP and BP are joined, and the figure is then creased along segments CP and DP, we obtain a triangular pyramid, all four of whose faces are isosceles triangles. Find the volume of this pyramid.



Solution:

After folding, A and B become one vertex X. Each half-diagonal of the rectangle has length $\frac{\sqrt{939}}{2}$. Thus

$$XP = CP = DP = \frac{\sqrt{939}}{2},$$

while $XC = XD = 13\sqrt{3}$ and $CD = 12\sqrt{3}$.

Place

$$C = (-6\sqrt{3}, 0, 0),$$
$$D = (6\sqrt{3}, 0, 0),$$

and $X = (0, \sqrt{399}, 0)$. These coordinates give the required lengths from X to C and D. Because P is equidistant from C and D, write $P = (0, u, h)$. Equating PC^2 and PX^2 gives

$$u = \frac{291}{2\sqrt{399}}.$$

Then $PC^2 = \frac{939}{4}$ yields

$$h^2 = \frac{507}{4} - u^2 = \frac{9801}{133},$$

so $h = \frac{99}{\sqrt{133}}$.

The base triangle XCD has area

$$\frac{1}{2}(12\sqrt{3})(\sqrt{399}) = 18\sqrt{133}.$$

Therefore the pyramid's volume is

$$\frac{1}{3}(18\sqrt{133})\left(\frac{99}{\sqrt{133}}\right) = 594.$$

15. Find $ax^5 + by^5$ if the real numbers $a, b, x,$ and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Solution:

Let $S_k = ax^k + by^k,\ p = x + y$, and $q = xy$. Since each of x, y satisfies $t^2 = pt - q$,

$$S_{k+2} = pS_{k+1} - qS_k.$$

Using $S_1 = 3,\ S_2 = 7,\ S_3 = 16$, and $S_4 = 42$ gives

$$\begin{aligned} 7p - 3q &= 16, \\ 16p - 7q &= 42. \end{aligned}$$

Solving yields $p = -14$ and $q = -38$. Therefore

$$\begin{aligned} S_5 &= pS_4 - qS_3 \\ &= -14(42) + 38(16) \\ &= 20. \end{aligned}$$

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