

1990 AIME

Time limit: 180 minutes

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1. The increasing sequence $2, 3, 5, 6, 7, 10, 11, \dots$ consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term of this sequence.

2. Find the value of

$$(52 + 6\sqrt{43})^{\frac{3}{2}} - (52 - 6\sqrt{43})^{\frac{3}{2}}.$$

3. Let P_1 be a regular r -gon and P_2 be a regular s -gon ($r \geq s \geq 3$) such that each interior angle of P_1 is $\frac{59}{58}$ as large as each interior angle of P_2 . What is the largest possible value of s ?

4. Find the positive solution to

$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0.$$

5. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $\frac{n}{75}$.
6. A biologist wants to calculate the number of fish in a lake. On May 1 she catches a random sample of 60 fish, tags them, and releases them. On September 1 she catches a random sample of 70 fish and finds that 3 of them are tagged. To calculate the number of fish in the lake on May 1, she assumes that 25% of these fish are no longer in the lake on September 1 (because of death and emigrations), that 40% of the fish were not in the lake May 1 (because of births and immigrations), and that the number of untagged fish and tagged fish in the September 1 sample are representative of the total population. What does the biologist calculate for the number of fish in the lake on May 1?

7. A triangle has vertices $P = (-8, 5)$, $Q = (-15, -19)$, and $R = (1, -7)$. The equation of the bisector of $\angle P$ can be written in the form $ax + 2y + c = 0$. Find $a + c$.
8. In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules:
- (1) The marksman first chooses a column from which a target is to be broken.
 - (2) The marksman must then break the lowest remaining target in the chosen column.
- If the rules are followed, in how many different orders can the eight targets be broken?
9. A fair coin is to be tossed 10 times. Let $\frac{i}{j}$, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i + j$.
10. The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C ?

11. Someone observed that $6! = 8 \cdot 9 \cdot 10$. Find the largest positive integer n for which $n!$ can be expressed as the product of $n - 3$ consecutive positive integers.

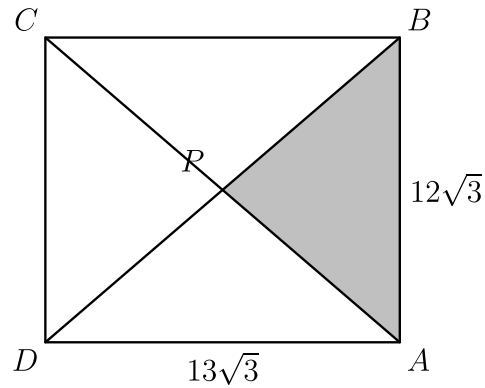
12. A regular 12-gon is inscribed in a circle of radius 12. The sum of the lengths of all sides and diagonals of the 12-gon can be written in the form

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6},$$

where a, b, c , and d are positive integers. Find $a + b + c + d$.

13. Let T be the set of powers 9^k , where k is an integer with $0 \leq k \leq 4000$. Given that 9^{4000} has 3817 digits and that its first (leftmost) digit is 9, how many elements of T have 9 as their leftmost digit?

- 14.** The rectangle $ABCD$ below has dimensions $AB = 12\sqrt{3}$ and $BC = 13\sqrt{3}$. Diagonals AC and BD intersect at P . If triangle ABP is cut out and removed, edges AP and BP are joined, and the figure is then creased along segments CP and DP , we obtain a triangular pyramid, all four of whose faces are isosceles triangles. Find the volume of this pyramid.



- 15.** Find $ax^5 + by^5$ if the real numbers a, b, x , and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Solutions: <https://live.poshenloh.com/past-contests/aime/1990/solutions>

