1989 AIME Solutions

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1. Compute $\sqrt{(31)(30)(29)(28)+1}$.

Solution:

We have $31 \cdot 28 = 868$ and $30 \cdot 29 = 870$. These products lie one below and one above 869, so their product is $869^2 - 1$. After the final 1 is added, the radicand is 869^2. Its positive square root is 869.

2. Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices?

Solution:

Every subset of at least three points determines exactly one convex polygon. There are $2^{10} = 1024$ subsets in all. Of these, $\binom{10}{0} = 1$, $\binom{10}{1} = 10$, and $\binom{10}{2} = 45$ have fewer than three points. Therefore the required number is $1024 - 1 - 10 - 45 = 968$.

3. Suppose n is a positive integer and d is a single digit in base 10. Find n if

$$\frac{n}{810} = 0.d25d25d25\ldots.$$

Solution:

The repeating decimal is $\frac{100d+25}{999}$. Thus

$$\begin{aligned} n &= \frac{810(100d+25)}{999} \\ &= \frac{750(4d+1)}{37}. \end{aligned}$$

Since $0 \le d \le 9$, the only way $37 \mid 4d+1$ is $4d+1 = 37$, so $d = 9$ and $n = 750$.

4. If $a < b < c < d < e$ are consecutive positive integers such that $b+c+d$ is a perfect square and $a+b+c+d+e$ is a perfect cube, what is the smallest possible value of c?

Solution:

The two sums are $3c$ and $5c$. If $c = \prod p^{v_p}$, then $3c$ being a square and $5c$ being a cube impose conditions on every exponent. For $p = 3$, the smallest exponent in c that is odd and divisible by 3 is 3. For $p = 5$, the smallest exponent that is even and congruent to $2 \pmod 3$ is 2. Every other prime exponent can be 0. Thus the least possible value is $c = 3^3 \cdot 5^2 = 675$.

5. When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let $\frac{i}{j}$, in lowest terms, be the probability that the coin comes up heads in exactly 3 out of 5 flips. Find $i + j$.

Solution:

Let p be the probability of heads. The condition gives

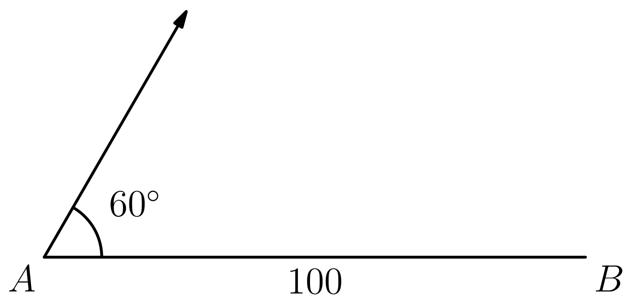
$$5p(1-p)^4 = 10p^2(1-p)^3.$$

The stated nonzero condition permits cancellation, yielding $1 - p = 2p$, so $p = \frac{1}{3}$. The probability of exactly three heads is

$$\binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{40}{243}.$$

Therefore $i + j = 40 + 243 = 283.$

6. Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a 60° angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?



Solution:

Suppose the skaters meet after t seconds. Their distances from A and B are $8t$ and $7t$, and the included angle at A is 60°. The Law of Cosines gives

$$\begin{aligned}(7t)^2 &= (8t)^2 + 100^2 \\ &\quad - 2(8t)(100)\cos 60^\circ.\end{aligned}$$

Hence $3t^2 - 160t + 2000 = 0$, whose roots are 20 and $\frac{100}{3}$. The earliest meeting occurs at $t = 20$, so Allie skates $8(20) = 160$ meters.

7. If the integer k is added to each of the numbers $36, 300,$ and $596,$ one obtains the squares of three consecutive terms of an arithmetic series. Find $k.$

Solution:

Let the three terms be $x,\ x+r,\ x+2r.$ Subtracting the square equations gives

$$r(2x+r) = 300 - 36 = 264$$

and

$$r(2x+3r) = 596 - 300 = 296.$$

Their difference is $2r^2 = 32,$ so the positive common difference is $r = 4.$ Then $4(2x+4) = 264,$ giving $x = 31.$ Therefore $k = x^2 - 36 = 961 - 36 = 925.$

8. Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned}
x_1 + 4x_2 + 9x_3 \\
+16x_4 + 25x_5 \\
+36x_6 + 49x_7 &= 1, \\
4x_1 + 9x_2 + 16x_3 \\
+25x_4 + 36x_5 \\
+49x_6 + 64x_7 &= 12, \\
9x_1 + 16x_2 + 25x_3 \\
+36x_4 + 49x_5 \\
+64x_6 + 81x_7 &= 123.
\end{aligned}$$

Find the value of

$$\begin{aligned}
16x_1 + 25x_2 + 36x_3 \\
+49x_4 + 64x_5 \\
+81x_6 + 100x_7.
\end{aligned}$$

Solution:

Define $S(t) = \sum_{i=1}^{7}(i+t)^2 x_i$. This is a quadratic polynomial in t, and the equations say $S(0) = 1,\ S(1) = 12$, and $S(2) = 123$. Its first two differences are 11 and 111, so the constant second difference is 100. The next first difference is therefore 211, giving $S(3) = 123 + 211 = 334$.

9. One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer n such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$

Find the value of n.

Solution:

Direct integer arithmetic gives

$$\begin{aligned} 133^5 &= 41{,}615{,}795{,}893, \\ 110^5 &= 16{,}105{,}100{,}000, \\ 84^5 &= 4{,}182{,}119{,}424, \\ 27^5 &= 14{,}348{,}907. \end{aligned}$$

Their sum is $61{,}917{,}364{,}224$. Repeated multiplication also gives $144^5 = 61{,}917{,}364{,}224$, so the positive integer n is 144.

10. Let a, b, c be the three sides of a triangle, and let α, β, γ be the angles opposite them. If $a^2 + b^2 = 1989c^2$, find

$$\frac{\cot \gamma}{\cot \alpha + \cot \beta}.$$

Solution:

First,

$$\begin{aligned}\cot \alpha + \cot \beta &= \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta} \\ &= \frac{\sin \gamma}{\sin \alpha \sin \beta}.\end{aligned}$$

Hence the desired ratio is

$$\frac{\cos \gamma \sin \alpha \sin \beta}{\sin^2 \gamma} = \frac{ab \cos \gamma}{c^2},$$

where the Law of Sines was used in the final equality. By the Law of Cosines,

$$\begin{aligned}2ab \cos \gamma &= a^2 + b^2 - c^2 \\ &= 1988c^2.\end{aligned}$$

Therefore the ratio is $\frac{1988}{2} = 994.$

11. A sample of 121 integers is given, each between 1 and 1000 inclusive, with repetitions allowed. The sample has a unique mode (most frequent value). Let D be the difference between the mode and the arithmetic mean of the sample. What is the largest possible value of $\lfloor |D| \rfloor$? (For real x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

Solution:

By reflecting every value x to $1001 - x$, it suffices to maximize the mean minus the mode. For a fixed modal frequency f, the extremal sample has f copies of 1, then fills the largest available integers with at most $f - 1$ copies each.

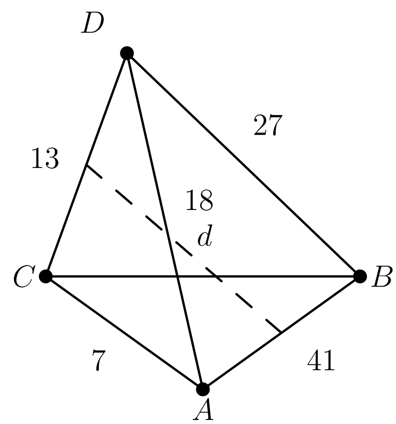
Put $121 - f = q(f-1) + r$, where $0 \le r < f - 1$. The nonmodal entries are $f - 1$ copies of each of $1000, 999, \ldots, 1001 - q$, followed by r copies of $1000 - q$. For $f = 2, 3, 4, 5, 6,$ this formula gives floors $924, 945, 947, 944, 939,$ respectively. If $f \ge 7,$ there are at most 114 nonmodal terms, so even the weaker bound $|D| \le \frac{114(999)}{121} < 942$ suffices. Thus the maximum occurs at $f = 4.$

The extremal sample contains four 1's and three copies of every integer from 962 through 1000. Put $T = 962 + 963 + \cdots + 1000$. Since $T = 38259,$

$$\begin{aligned} |D| &= \frac{3T + 4}{121} - 1 \\ &= \frac{114660}{121} \\ &= 947 + \frac{73}{121}. \end{aligned}$$

Therefore the largest possible floor is $947.$

12. Let $ABCD$ be a tetrahedron with $AB = 41, AC = 7, AD = 18, BC = 36, BD = 27,$ and $CD = 13,$ as shown in the figure. Let d be the distance between the midpoints of edges AB and $CD.$ Find $d^2.$



Solution:

Let the vertex names also denote their position vectors. The vector between the midpoints is $\frac{A+B-C-D}{2}$. Expanding squared lengths gives

$$\begin{aligned} 4d^2 = AC^2 + AD^2 \\ + BC^2 + BD^2 \\ - AB^2 - CD^2. \end{aligned}$$

Therefore

$$\begin{aligned} 4d^2 = 7^2 + 18^2 + 36^2 + 27^2 \\ - 41^2 - 13^2 = 548, \end{aligned}$$

so $d^2 = 137.$

13. Let S be a subset of $\{1, 2, 3, \ldots, 1989\}$ such that no two members of S differ by 4 or 7. What is the largest number of elements S can have?

Solution:

On any eleven consecutive integers, join two numbers when their difference is 4 or 7. Because $7 = 11 - 4$, this graph is an 11-cycle, whose largest independent set has size 5. Similarly, any ten consecutive integers induce a path on ten vertices and contribute at most 5. Since $1989 = 179 \cdot 11 + 2 \cdot 10$, this gives $|S| \leq 179(5) + 2(5) = 905$.

This bound is attained by taking every integer congruent to $1,\ 3,\ 4,\ 6,$ or $9 \pmod{11}$. No two selected residues differ by 4 or 7 modulo 11, and each of these five residues occurs 181 times from 1 through 1989. Thus the maximum is $5(181) = 905$.

14. Given a positive integer n, it can be shown that every complex number of the form $r + si$, where r and s are integers, can be uniquely expressed in the base $-n + i$ using the integers $0, 1, \ldots, n^2$ as digits. That is, the equation

$$\begin{aligned} r + si = a_m(-n+i)^m & \\ + a_{m-1}(-n+i)^{m-1} & \\ + \cdots + a_1(-n+i) & \\ + a_0 & \end{aligned}$$

is true for a unique choice of nonnegative integer m and digits $a_0, a_1, \ldots, a_m$ chosen from the set $\{0, 1, 2, \ldots, n^2\}$, with $a_m \neq 0$. We write

$$r + si = (a_m a_{m-1} \ldots a_1 a_0)_{-n+i}$$

to denote the base $-n + i$ expansion of $r + si$. There are only finitely many integers $k + 0i$ that have four-digit expansions

$$k = (a_3 a_2 a_1 a_0)_{-3+i} \qquad a_3 \neq 0.$$

Find the sum of all such k.

Solution:

Let $b = -3 + i$. Then $b^2 = 8 - 6i$ and $b^3 = -18 + 26i$.

The imaginary part of $a_3b^3 + a_2b^2 + a_1b + a_0$ is

$$26a_3 - 6a_2 + a_1.$$

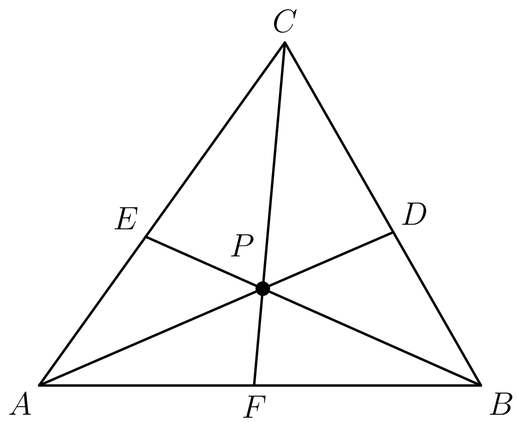
Thus $a_1 = 6a_2 - 26a_3$. With $1 \le a_3 \le 9$ and $0 \le a_1, a_2 \le 9$, the only possibilities are

$$(a_3, a_2, a_1) = (1, 5, 4),$$
$$(a_3, a_2, a_1) = (2, 9, 2).$$

The corresponding real parts are $10 + a_0$ and $30 + a_0$, respectively. As a_0 ranges from 0 through 9, the required sum is

$$\begin{aligned} &(10 + 11 + \cdots + 19) \\ &\quad + (30 + 31 + \cdots + 39) \\ &= 145 + 345 = 490. \end{aligned}$$

15. Point P is inside triangle ABC. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB (see the figure). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of triangle ABC.



Solution:

Since $AP = PD$, the barycentric weight of A at P is $\frac{1}{2}$. Since $BP : PE = 3 : 1$, the weight of B is $\frac{1}{4}$, so the weight of C is also $\frac{1}{4}$. Along CF, this means $\frac{PF}{CF} = \frac{1}{4}$, hence $PF = 5$ and $CP = 15$.

Place P at the origin and denote the position vectors of A, B, C by the same letters. The barycentric relation is $2A + B + C = 0$. Thus $C = -2A - B$. Using $|A| = 6$, $|B| = 9$, and $|C| = 15$,

$$\begin{aligned} 225 &= |2A + B|^2 \\ &= 4(36) + 81 + 4A \cdot B, \end{aligned}$$

so $A \cdot B = 0$. Therefore $PA \perp PB$. Expanding the cross product gives $(B - A) \times (C - A) = 4(A \times B)$. Consequently,

$$\begin{aligned} [ABC] &= 2|A \times B| \\ &= 2(6)(9) = 108. \end{aligned}$$

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