1988 AIME Solutions

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1. One commercially available ten-button lock may be opened by depressing—in any order—the correct five buttons. One sample has $\{1, 2, 3, 6, 9\}$ as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?

Solution:

The redesigned lock permits every nonempty subset except the full set, giving $2^{10} - 2 = 1022$ combinations. The original lock permits $\binom{10}{5} = 252$. Thus the number of additional combinations is $1022 - 252 = 770$.

2. For any positive integer k, let $f_1(k)$ denote the square of the sum of the digits of k. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1988}(11)$.

Solution:

The iterates begin $4,\ 16,\ 49,\ 169,\ 256$, followed by $169,\ 256, \ldots$. Thus from the fourth iterate onward, even indices give 169 and odd indices give 256. Since 1988 is even, $f_{1988}(11) = 169$.

3. Find $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$.

Solution:

Put $y = \log_2 x > 0$. Since $\log_8 x = \frac{y}{3}$, the equation becomes

$$\log_2(\frac{y}{3}) = \frac{1}{3}\log_2 y.$$

Hence $\frac{2}{3}\log_2 y = \log_2 3$, so $y = 3^{\frac{3}{2}} = 3\sqrt{3}$. Therefore $(\log_2 x)^2 = y^2 = 27$.

4. Suppose that $|x_i| < 1$ for $i = 1, 2, \ldots, n$. Suppose further that

$$\begin{aligned} &|x_1| + |x_2| + \cdots + |x_n| \\ &= 19 + \big|x_1 + x_2 \\ &\qquad + \cdots + x_n\big|. \end{aligned}$$

What is the smallest possible value of n?

Solution:

Let P be the sum of the positive x_i and N the sum of the absolute values of the negative x_i. Then

$$\begin{aligned} P + N - |P - N| &= 2\min(P, N) \\ &= 19, \end{aligned}$$

so both P and N are at least 9.5. Because every $|x_i| < 1$, each sign requires at least 10 terms, giving $n \geq 20$. Equality is attainable with ten terms equal to 0.95 and ten equal to -0.95, so the minimum is 20.

5. Let $\frac{m}{n}$, in lowest terms, be the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}. Find $m+n$.

Solution:

Every divisor is $2^a 5^b$ with $0 \leq a, b \leq 99$, so there are $100^2 = 10000$ divisors. A multiple of 10^{88} requires $88 \leq a, b \leq 99$, giving $12^2 = 144$ divisors. The probability is $\frac{144}{10000} = \frac{9}{625}$, and $m+n = 9+625 = 634$.

6. It is possible to place positive integers into the vacant twenty-one squares of the 5×5 square shown below so that the numbers in each row and column form arithmetic sequences. Find the number that must occupy the vacant square marked by the asterisk $(*)$.

			$*$	
	74			
				186
		103		
0				

Solution:

Index rows and columns from 0 to 4. The row and column conditions give the general entry $A + Bi + Cj + Dij$. The four shown values yield

$$\begin{aligned} A + 4B &= 0, \\ A + B + C + D &= 74, \\ A + 2B + 4C + 8D &= 186, \\ A + 3B + 2C + 6D &= 103. \end{aligned}$$

Solving gives $A = 52, B = -13, C = 30$, and $D = 5$. The asterisk is at $(i, j) = (0, 3)$, so its value is $A + 3C = 52 + 90 = 142$.

7. In triangle $ABC, \tan \angle CAB = \frac{22}{7}$, and the altitude from A divides BC into segments of length 3 and 17. What is the area of triangle ABC?

Solution:

Let the altitude length be h. From A, vectors to the endpoints of BC may be taken as $(-3, -h)$ and $(17, -h)$. Therefore

$$\tan \angle CAB = \frac{20h}{h^2 - 51} = \frac{22}{7}.$$

Thus $11h^2 - 70h - 561 = 0$, whose positive root is $h = 11$. Since $BC = 3 + 17 = 20$, the area is $\frac{1}{2}(20)(11) = 110$.

8. The function f, defined on the set of ordered pairs of positive integers, satisfies the following properties:

$$f(x,x) = x,$$
$$f(x,y) = f(y,x),$$
$$(x+y)f(x,y) = yf(x,x+y).$$

Calculate $f(14,52)$.

Solution:

For $y > x$, the third property applied to $(x, y-x)$ gives

$$f(x,y) = \frac{y}{y-x} f(x, y-x).$$

The least common multiple obeys the identical relation, because $\gcd(x,y) = \gcd(x, y-x)$. Hence the quotient $\frac{f(x,y)}{\operatorname{lcm}(x,y)}$ is unchanged by each subtraction step of the Euclidean algorithm. On the diagonal it equals $\frac{f(g,g)}{g} = 1$, so $f(x,y) = \operatorname{lcm}(x,y)$. Therefore

$$\begin{aligned} f(14,52) &= \frac{14 \cdot 52}{\gcd(14,52)} \\ &= 364. \end{aligned}$$

9. Find the smallest positive integer whose cube ends in 888.

Solution:

Write the number as $10a+2$. Expanding the required congruence modulo 125 and dividing by 5 gives

$$20a^2 - a - 1 \equiv 0 \pmod{25}.$$

Reducing first modulo 5 gives $a \equiv 4 \pmod 5$; substituting $a = 4 + 5b$ then gives $b \equiv 3 \pmod 5$. Thus $a \equiv 19 \pmod{25}$, and the smallest candidate is $10(19) + 2 = 192$. It is even, so its cube is also $0 \pmod 8$, and indeed $192^3 = 7{,}077{,}888$.

10. A convex polyhedron has for its faces 12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face?

Solution:

The total number of face-vertex incidences is $12(4) + 8(6) + 6(8) = 144$. Three faces meet at each vertex, so $V = 48$. The same sum counts each edge twice, so $E = 72$.

There are $\binom{48}{2} = 1128$ vertex pairs. Summing pairs on faces gives $12\binom{4}{2} + 8\binom{6}{2} + 6\binom{8}{2} = 360$. Every edge was counted twice in this sum and every other same-face pair once, so the number of distinct boundary pairs is $360 - E = 288$. Hence $1128 - 288 = 840$ joining segments lie in the interior.

11. Let $w_1, w_2, \ldots, w_n$ be complex numbers. A line L in the complex plane is called a mean line for the points $w_1, w_2, \ldots, w_n$ if L contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that

$$\sum_{k=1}^{n} (z_k - w_k) = 0.$$

For the numbers $w_1 = 32 + 170i, w_2 = -7 + 64i, w_3 = -9 + 200i, w_4 = 1 + 27i,$ and $w_5 = -14 + 43i,$ there is a unique mean line with y-intercept $3.$ Find the slope of this mean line.

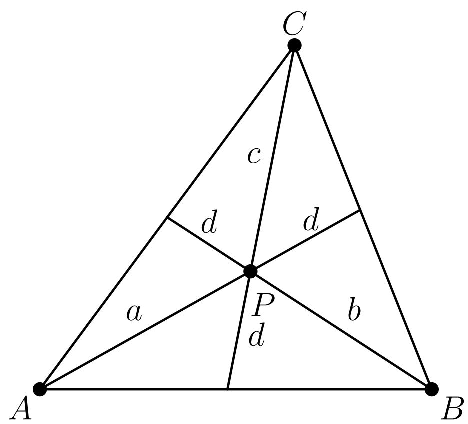
Solution:

The condition says that the average of the z_k equals the average of the w_k. Since all z_k lie on L, their average lies on L; conversely, any line through the average works by taking all z_k equal to that point. The centroid satisfies

$$\begin{aligned}\bar{x} &= \frac{32 - 7 - 9 + 1 - 14}{5} = \frac{3}{5},\\ \bar{y} &= \frac{170 + 64 + 200 + 27 + 43}{5}\\ &= \frac{504}{5}.\end{aligned}$$

The line through this point and $(0, 3)$ has slope $\frac{\left(\frac{504}{5} - 3\right)}{\frac{3}{5}} = 163.$

12. Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let $a, b, c,$ and d denote the lengths of the segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$.



Solution:

Along the cevian from A, the barycentric coordinate at A is $\frac{d}{a+d}$, and similarly at the other vertices. Since the three coordinates sum to 1 and $d = 3$,

$$\frac{3}{a+3} + \frac{3}{b+3} + \frac{3}{c+3} = 1.$$

Put $s_1 = a + b + c = 43$, $s_2 = ab + bc + ca$, and $s_3 = abc$. Clearing denominators gives

$$3(s_2 + 6s_1 + 27) = s_3 + 3s_2 + 9s_1 + 27.$$

Therefore $s_3 = 9s_1 + 54$, so $s_3 = 9(43) + 54 = 441$.

13. Find a if a and b are integers such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.

Solution:

Modulo $x^2 - x - 1,$ the relation $x^2 = x + 1$ gives $x^n = F_n x + F_{n-1}$. Hence the remainder is

$$\begin{aligned} r(x) = {} & (1597a + 987b)x \\ & + (987a + 610b + 1). \end{aligned}$$

Both coefficients must vanish. The determinant of the coefficient matrix is

$$1597(610) - 987^2 = 1$$

by Cassini's identity. Cramer's rule then gives $a = 987$.

14. Let C be the graph of $xy = 1$, and denote by C^* the reflection of C in the line $y = 2x$. Let the equation of C^* be written in the form

$$12x^2 + bxy + cy^2 + d = 0.$$

Find the product bc.

Solution:

Reflection in the line spanned by $(1, 2)$ has matrix

$$\frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

Thus a point (x, y) on C^* came from $u = \frac{-3x+4y}{5}$ and $v = \frac{4x+3y}{5}$ on C. Substituting $uv = 1$ gives

$$-12x^2 + 7xy + 12y^2 - 25 = 0,$$

or $12x^2 - 7xy - 12y^2 + 25 = 0$. Hence $b = -7,\ c = -12$, and $bc = 84$.

15. In an office at various times during the day, the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in-box. When there is time, the secretary takes the top letter off the pile and types it. There are nine letters to be typed during the day, and the boss delivers them in the order $1, 2, 3, 4, 5, 6, 7, 8, 9$.

While leaving for lunch, the secretary tells a colleague that letter 8 has already been typed, but says nothing else about the morning's typing. The colleague wonders which of the nine letters remain to be typed after lunch and in what order they will be typed. Based upon the above information, how many such after-lunch typing orders are possible? (That there are no letters left to be typed is one of the possibilities.)

Solution:

Any subset of $1, \ldots, 7$ can remain in the stack after 8 has been typed, and those remaining letters must later be typed in decreasing order. If 9 was already typed, choosing that subset gives $2^7 = 128$ possible orders.

If 9 remains, choose k of the seven smaller letters and insert 9 into any of the $k+1$ positions in their decreasing order. This gives

$$\begin{aligned}\sum_{k=0}^{7} \binom{7}{k}(k+1) &= 7 \cdot 2^6 + 2^7 \\ &= 576.\end{aligned}$$

Every such order can be realized by suitable morning choices, so the total is $128 + 576 = 704$.

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