1987 AIME Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/aime/1987/solutions

1. An ordered pair (m, n) of nonnegative integers is called "simple" if adding $m + n$ in base 10 requires no carrying. Find the number of simple ordered pairs that sum to 1492.

Solution:

For a target digit d, there are $d + 1$ ordered pairs of nonnegative digits with sum d. The four digits $1, 4, 9, 2$ therefore give $2, 5, 10, 3$ choices independently. The answer is $2 \cdot 5 \cdot 10 \cdot 3 = 300$.

2. What is the largest possible distance between two points, one on the sphere of radius 19 centered at $(-2, -10, 5)$ and the other on the sphere of radius 87 centered at $(12, 8, -16)$?

Solution:

The squared distance between the centers is $14^2 + 18^2 + (-21)^2 = 961$, so they are $\sqrt{961} = 31$ apart. By the triangle inequality, the greatest point-to-point distance is obtained on the line of centers and equals $19 + 31 + 87 = 137$.

3. A proper divisor of a natural number is a positive integral divisor other than 1 and the number itself. A natural number greater than 1 is called "nice" if it equals the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

Solution:

If N has d positive divisors, their product is $N^{\frac{d}{2}}$. Removing 1 and N leaves product $N^{\frac{d}{2}-1}$, which equals N exactly when $d=4$. Thus nice numbers are precisely p^3 and pq for distinct primes. The first ten are $6, 8, 10, 14, 15, 21, 22, 26, 27, 33,$ whose sum is $182.$

4. Find the area of the region enclosed by the graph of $|x-60|+|y|=|\frac{x}{4}|$.

Solution:

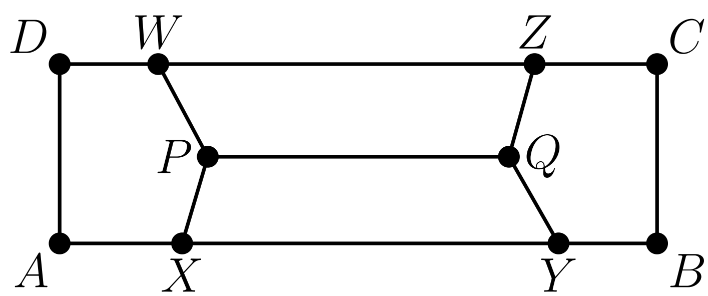
We need $|y|=\frac{|x|}{4}-|x-60|\geq 0$. This forces $48\leq x\leq 80$. At $x=48, 60, 80,$ the boundary values are respectively $y=0,\ y=\pm 15,\ y=0$. Hence the region is a kite with perpendicular diagonals $80-48=32$ and 30, so its area is $\frac{1}{2}(32)(30)=480.$

5. Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

Solution:

Rearranging gives $(3x^2+1)(y^2-10) = 507$, where $507 = 3 \cdot 13^2$. The positive divisors of 507 congruent to $1 \pmod 3$ are $1, 13, 169$. Setting $3x^2+1$ equal to these shows that only 13 gives an integral square $x^2 = 4$; then $y^2 - 10 = 39$, so $y^2 = 49$. Therefore $3x^2y^2 = 3 \cdot 4 \cdot 49 = 588$.

6. Rectangle $ABCD$ is divided into four parts of equal area by five segments as shown, where $XY = YB + BC + CZ$, $YB + BC + CZ = ZW$, $ZW = WD + DA + AX$, and $PQ \parallel AB$. Find AB (in cm) if $BC = 19$ cm and $PQ = 87$ cm.



Solution:

Put $AB = L$ and let the four equal boundary lengths be t. Writing $AX = x$ and $DW = w$, the left boundary condition gives $x + w = t - 19$. Substituting $XY = ZW = t$ into the right condition gives $2L + 38 = 4t$, so $t = \frac{L+19}{2}$.

The upper and lower central regions are trapezoids with the same bases t and 87. Since their areas are equal, PQ lies halfway up the 19-cm rectangle. Each central region therefore has area $\frac{19(t+87)}{4}$. This is one quarter of the rectangle, $\frac{19L}{4}$, so $t + 87 = L$. Combining with $t = \frac{L+19}{2}$ yields $L = 193$.

7. Let $[r, s]$ denote the least common multiple of positive integers r, s. Find the number of ordered triples (a, b, c) for which $[a, b] = 1000$, $[b, c] = 2000$, and $[c, a] = 2000$.

Solution:

For the exponent of 5, all three pairwise maxima equal 3. Thus at least two exponents are 3 : there is one all-3 triple and $3 \cdot 3$ triples with exactly two 3's, for 10 choices.

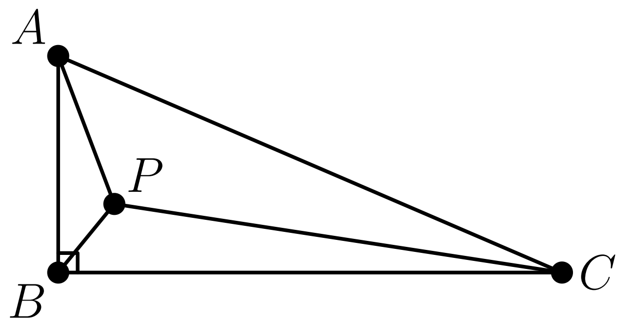
For the exponent of 2, the first pair has maximum 3 while the other two have maximum 4. The exponent of c must be 4, and the exponents of a, b lie in $\{0, 1, 2, 3\}$ with maximum 3, giving $4^2 - 3^2 = 7$ choices. Independence gives $10 \cdot 7 = 70$.

8. What is the largest positive integer n for which there is a unique integer k such that $\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}$?

Solution:

The inequalities are equivalent to $\frac{6n}{7} < k < \frac{7n}{8}$. This interval has length $\frac{n}{56}$. At $n = 112$ it is $(96, 98)$, containing only 97. For $n > 112$ its length exceeds 2, so it contains at least two integers. Thus the largest possible n is 112.

9. Triangle ABC has a right angle at B and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.



Solution:

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the vectors from P to A, B, C, and put $|\mathbf{c}| = x$. Their pairwise angles are 120°, so $\mathbf{a} \cdot \mathbf{b} = -30,\ \mathbf{b} \cdot \mathbf{c} = -3x$, and $\mathbf{a} \cdot \mathbf{c} = -5x$. Since $AB \perp BC$, $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b}) = 0$. Expanding gives $-5x + 30 + 3x + 36 = 0$, so $x = 33$.

10. Al walks down an escalator that is moving up and counts 150 steps. Bob walks up and counts 75 steps. If Al's walking speed is three times Bob's, how many steps are visible at a given time? Assume this is constant.

Solution:

Bob's time is $\frac{75}{b}$, so $N = (b + e)(\frac{75}{b})$, or $N = 75(1 + \frac{e}{b})$. Al's time is $\frac{150}{3b} = \frac{50}{b}$, so $N = (3b - e)(\frac{50}{b})$, or $N = 150 - \frac{50e}{b}$. Equating gives $\frac{125e}{b} = 75$, hence $\frac{e}{b} = \frac{3}{5}$ and $N = 75(1 + \frac{3}{5}) = 120$.

11. Find the largest possible k for which 3^{11} is expressible as the sum of k consecutive positive integers.

Solution:

If the first term is a, then $2 \cdot 3^{11} = k(2a + k - 1)$. Thus k is 3^j or $2 \cdot 3^j$. The largest viable even choice is $k = 2 \cdot 3^5 = 486$, for which $2a + k - 1 = 3^6 = 729$ and $a = 122 > 0$. The next candidates, 3^6 and $2 \cdot 3^6$, force a nonpositive first term, as do all larger choices. Hence $k = 486$.

12. Let m be the smallest integer whose cube root has the form $n + r$, where n is a positive integer and $0 < r < \frac{1}{1000}$. Find n.

Solution:

For a given n, the closest integer cube-root candidate above n is $m = n^3 + 1$. We need $n^3 + 1 < (n + 0.001)^3$, or $1 < 0.003n^2 + 0.000003n + 10^{-9}$. This fails at $n = 18$ and holds at $n = 19$. Smaller n cannot work with a larger m, so the smallest m occurs with $n = 19$.

13. A given sequence $r_1, r_2, \ldots, r_n$ of distinct real numbers can be put in ascending order by means of one or more "bubble passes." A bubble pass through a given sequence consists of comparing the second term with the first term, and exchanging them if and only if the second term is smaller, then comparing the third term with the second term and exchanging them if and only if the third term is smaller, and so on in order, through comparing the last term, r_n, with its current predecessor and exchanging them if and only if the last term is smaller.

The example below shows how the sequence $1, 9, 8, 7$ is transformed into the sequence $1, 8, 7, 9$ by one bubble pass. The numbers compared at each step are underlined.

$$\begin{array}{cccc}
\underline{1} & \underline{9} & 8 & 7 \\
1 & \underline{9} & \underline{8} & 7 \\
1 & 8 & \underline{9} & \underline{7} \\
1 & 8 & 7 & 9
\end{array}$$

Suppose that $n = 40$, and that the terms of the initial sequence $r_1, r_2, \ldots, r_{40}$ are distinct from one another and are in random order. Let $\frac{p}{q}$, in lowest terms, be the probability that the number that begins as r_{20} will end up, after one bubble pass, in the 30th place. Find $p + q$.

Solution:

For r_{20} to move right through position 30, it must exceed every other term among $r_1, \ldots, r_{30}$. It stops at position 30 exactly when $r_{31} > r_{20}$. Thus among the first 31 terms, r_{31} must be greatest and r_{20} second greatest. These two ordered rank assignments have probability $\frac{1}{31} \cdot \frac{1}{30} = \frac{1}{930}$. Therefore $p + q = 1 + 930 = 931$.

14. Compute

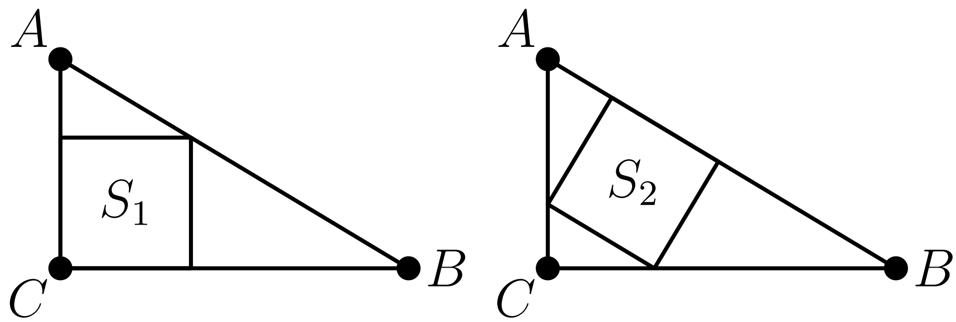
$$\frac{\begin{gathered}(10^4+324)(22^4+324)\\ \cdot(34^4+324)(46^4+324)\\ \cdot(58^4+324)\end{gathered}}{\begin{gathered}(4^4+324)(16^4+324)\\ \cdot(28^4+324)(40^4+324)\\ \cdot(52^4+324)\end{gathered}}.$$

Solution:

Let $g(x)=x^2+6x+18$. Sophie Germain's identity gives $x^4+324=g(x-6)g(x)$. The numerator therefore supplies $g(4), g(10), \ldots, g(58)$, while the denominator supplies $g(-2), g(4), \ldots, g(52)$. Everything cancels except

$$\begin{aligned}\frac{g(58)}{g(-2)} &= \frac{58^2+6(58)+18}{4-12+18}\\ &= \frac{3730}{10} = 373.\end{aligned}$$

15. Squares S_1, S_2 are inscribed in right triangle ABC as shown. Find $AC + CB$ if $\text{area}(S_1) = 441$ and $\text{area}(S_2) = 440$.



Solution:

Put $p = a + b,\ q = ab$, and let the hypotenuse be c. Since S_1 has side 21, the standard leg-aligned-square relation gives $21 = \frac{ab}{a+b}$, so $q = 21p$. Hence $c^2 = p^2 - 2q = p(p-42)$.

The altitude to the hypotenuse is $h = \frac{q}{c}$. If the side of S_2 is t, similarity gives $t = \frac{ch}{c+h}$. Substitution simplifies this to $t = \frac{21c}{p-21}$. Therefore

$$440 = \frac{441p(p-42)}{(p-21)^2}.$$

Since $p(p-42) = (p-21)^2 - 441$, this becomes

$$440 = 441\left(1 - \frac{441}{(p-21)^2}\right).$$

Thus $(p-21)^2 = 441^2$, and $p > 42$ gives $p = 462$.

Problems: https://live.poshenloh.com/past-contests/aime/1987