1987 AIME

Time limit: 180 minutes

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1. An ordered pair (m, n) of nonnegative integers is called "simple" if adding $m + n$ in base 10 requires no carrying. Find the number of simple ordered pairs that sum to 1492.

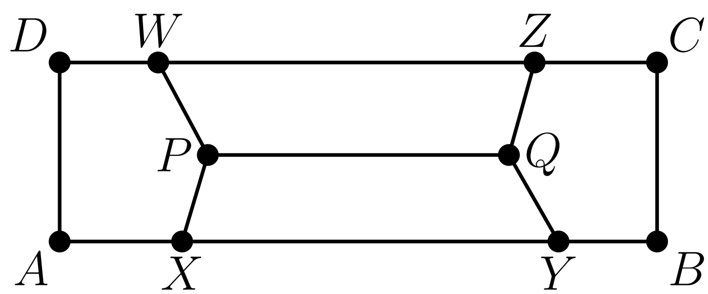
2. What is the largest possible distance between two points, one on the sphere of radius 19 centered at $(-2, -10, 5)$ and the other on the sphere of radius 87 centered at $(12, 8, -16)$?

3. A proper divisor of a natural number is a positive integral divisor other than 1 and the number itself. A natural number greater than 1 is called "nice" if it equals the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

4. Find the area of the region enclosed by the graph of $|x-60|+|y|=\left|\frac{x}{4}\right|$.

5. Find $3x^2y^2$ if x and y are integers such that $y^2+3x^2y^2=30x^2+517$.

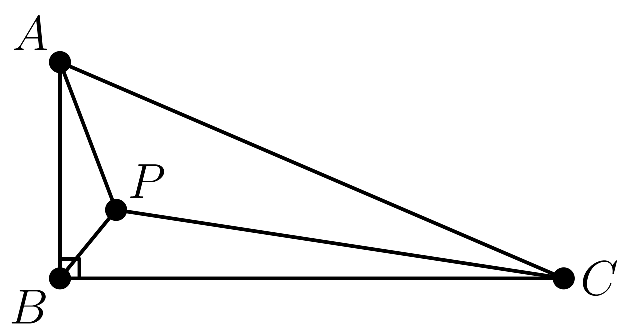
6. Rectangle $ABCD$ is divided into four parts of equal area by five segments as shown, where $XY=YB+BC+CZ, YB+BC+CZ=ZW, ZW=WD+DA+AX$, and $PQ \parallel AB$. Find AB (in cm) if $BC=19$ cm and $PQ=87$ cm.



7. Let $[r,s]$ denote the least common multiple of positive integers r,s. Find the number of ordered triples (a,b,c) for which $[a,b]=1000$, $[b,c]=2000$, and $[c,a]=2000$.

8. What is the largest positive integer n for which there is a unique integer k such that $\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}$?

9. Triangle ABC has a right angle at B and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.



10. Al walks down an escalator that is moving up and counts 150 steps. Bob walks up and counts 75 steps. If Al's walking speed is three times Bob's, how many steps are visible at a given time? Assume this is constant.

11. Find the largest possible k for which 3^{11} is expressible as the sum of k consecutive positive integers.

12. Let m be the smallest integer whose cube root has the form $n+r$, where n is a positive integer and $0<r<\frac{1}{1000}$. Find n.

13. A given sequence $r_1, r_2, \ldots, r_n$ of distinct real numbers can be put in ascending order by means of one or more "bubble passes." A bubble pass through a given sequence consists of comparing the second term with the first term, and exchanging them if and only if the second term is smaller, then comparing the third term with the second term and exchanging them if and only if the third term is smaller, and so on in order, through comparing the last term, r_n, with its current predecessor and exchanging them if and only if the last term is smaller.

The example below shows how the sequence $1, 9, 8, 7$ is transformed into the sequence $1, 8, 7, 9$ by one bubble pass. The numbers compared at each step are underlined.

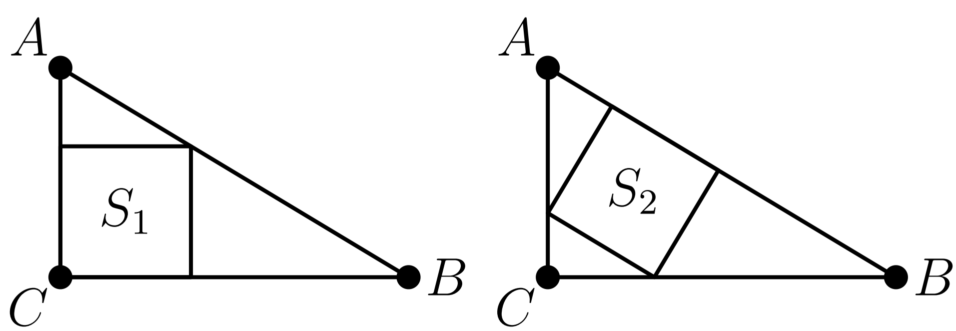
$$\begin{array}{cccc}
\underline{1} & \underline{9} & 8 & 7 \\
1 & \underline{9} & \underline{8} & 7 \\
1 & 8 & \underline{9} & \underline{7} \\
1 & 8 & 7 & 9
\end{array}$$

Suppose that $n=40$, and that the terms of the initial sequence $r_1, r_2, \ldots, r_{40}$ are distinct from one another and are in random order. Let $\frac{p}{q}$, in lowest terms, be the probability that the number that begins as r_{20} will end up, after one bubble pass, in the 30th place. Find $p+q$.

14. Compute

$$\frac{\begin{gathered}(10^4+324)(22^4+324)\\ \cdot(34^4+324)(46^4+324)\\ \cdot(58^4+324)\end{gathered}}{\begin{gathered}(4^4+324)(16^4+324)\\ \cdot(28^4+324)(40^4+324)\\ \cdot(52^4+324)\end{gathered}}.$$

15. Squares S_1, S_2 are inscribed in right triangle ABC as shown. Find $AC + CB$ if $\mathrm{area}(S_1) = 441$ and $\mathrm{area}(S_2) = 440$.



Solutions: https://live.poshenloh.com/past-contests/aime/1987/solutions