1986 AIME Solutions

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1. What is the sum of the solutions to the equation $\sqrt[4]{x} = \frac{12}{7-\sqrt[4]{x}}$?

Solution:

Let $t = \sqrt[4]{x}$, so $t \geq 0$. The equation becomes

$$t(7-t) = 12,$$

or $t^2 - 7t + 12 = 0$. Thus $t = 3$ or $t = 4$, and both values are valid in the original equation. Hence $x = 3^4 = 81$ or $x = 4^4 = 256$, and their sum is $81 + 256 = 337$.

2. Evaluate the product

$$
\begin{gathered}
(\sqrt{5}+\sqrt{6}+\sqrt{7}) \\
\cdot(-\sqrt{5}+\sqrt{6}+\sqrt{7}) \\
\cdot(\sqrt{5}-\sqrt{6}+\sqrt{7}) \\
\cdot(\sqrt{5}+\sqrt{6}-\sqrt{7}).
\end{gathered}
$$

Solution:

Pair the first two factors, then the last two:

$$
\begin{gathered}
(\sqrt{6}+\sqrt{7})^2-(\sqrt{5})^2 \\
=8+2\sqrt{42}, \\
(\sqrt{5})^2-(\sqrt{7}-\sqrt{6})^2 \\
=-8+2\sqrt{42}.
\end{gathered}
$$

Therefore the requested product is

$$
\begin{gathered}
(8+2\sqrt{42})(-8+2\sqrt{42}) \\
=168-64=104.
\end{gathered}
$$

3. If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, what is $\tan(x+y)$?

Solution:

Since

$$\cot x + \cot y = \frac{\tan x + \tan y}{\tan x \tan y},$$

the given equations yield $\tan x \tan y = \frac{25}{30} = \frac{5}{6}$. Therefore

$$\begin{aligned} \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ &= \frac{25}{1 - \frac{5}{6}} \\ &= 150. \end{aligned}$$

4. Determine $3x_4 + 2x_5$ if $x_1, x_2, x_3, x_4,$ and x_5 satisfy the system

$$\begin{aligned}2x_1 + x_2 + x_3 + x_4 + x_5 &= 6,\\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12,\\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24,\\ x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48,\\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96.\end{aligned}$$

Solution:

Put $S = x_1 + x_2 + x_3 + x_4 + x_5$. The five equations say that $S + x_i$ equals $6, 12, 24, 48, 96,$ respectively. Adding them gives

$$\begin{aligned}6S &= 6 + 12 + 24 + 48 + 96\\ &= 186,\end{aligned}$$

so $S = 31$. Hence $x_4 = 48 - 31 = 17$ and $x_5 = 96 - 31 = 65$. The requested value is $3(17) + 2(65) = 181$.

5. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

Solution:

Modulo $n + 10$, we have $n \equiv -10$. Thus

$$\begin{aligned} n^3 + 100 &\equiv (-10)^3 + 100 \\ &\equiv -900 \pmod{n + 10}. \end{aligned}$$

The condition is therefore equivalent to $n + 10 \mid 900$. Since n is positive, $n + 10$ is a positive divisor of 900, and its largest possible value is 900. This gives $n = 900 - 10 = 890$.

6. The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in an incorrect sum of 1986. What was the number of the page that was added twice?

Solution:

Consecutive triangular numbers around 1986 are

$$\begin{aligned} \frac{62 \cdot 63}{2} &= 1953, \\ \frac{63 \cdot 64}{2} &= 2016. \end{aligned}$$

Hence the book has 62 pages, and the extra summand is $1986 - 1953 = 33$, which is indeed a valid page number.

7. The increasing sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term of this sequence.

Solution:

A sum of distinct powers of 3 has only 0's and 1's in base 3. As these strings increase, they occur in the same order as binary numerals with the same digit strings. Since

$$100 = 1100100_2,$$

the 100th positive term is

$$\begin{aligned} 1100100_3 &= 3^6 + 3^5 + 3^2 \\ &= 729 + 243 + 9 \\ &= 981. \end{aligned}$$

8. Let S be the sum of the base 10 logarithms of all the proper divisors of 1000000. What is the integer nearest to S?

Solution:

Let $N = 1000000 = 2^6 5^6$. It has $(6+1)(6+1) = 49$ positive divisors. The product of all of them is $N^{\frac{49}{2}}$, so the sum of their base-10 logarithms is

$$\frac{49}{2} \log_{10} N = \frac{49}{2} \cdot 6 = 147.$$

The proper divisors include 1 but exclude N itself. Subtracting $\log_{10} N = 6$ gives $S = 141$, already an integer.

9. In $\triangle ABC$, $AB = 425$, $BC = 450$, and $AC = 510$. An interior point P is drawn, and segments are drawn through P parallel to the sides of the triangle. If these three segments have equal length d, find d.

Solution:

Write $a = BC = 450$, $b = CA = 510$, and $c = AB = 425$. Let x, y, z be the distances from P to BC, CA, AB, respectively, each divided by the corresponding altitude. Area decomposition gives $x + y + z = 1$.

By similar triangles, the segment through P parallel to BC has length $a(1 - x)$, and similarly the other two lengths are $b(1 - y)$ and $c(1 - z)$. Since all three equal d,

$$\begin{aligned} x &= 1 - \frac{d}{a}, \\ y &= 1 - \frac{d}{b}, \\ z &= 1 - \frac{d}{c}. \end{aligned}$$

Their sum is 1, so

$$d = \frac{2abc}{ab + bc + ca}.$$

Substituting the three side lengths gives

$$\begin{aligned} d &= \frac{195075000}{637500} \\ &= 306. \end{aligned}$$

10. In a parlor game, the magician asks one of the participants to think of a three-digit number (abc), where $a, b,$ and c represent base-10 digits in the indicated order. The magician then asks this person to form the numbers $(acb), (bca), (bac), (cab),$ and (cba), to add these five numbers, and to reveal their sum N. If told N, the magician can identify the original number (abc). Play the role of the magician and determine (abc) if $N = 3194.$

Solution:

Across all six permutations, each digit occurs twice in each place, so their total is $222(a+b+c)$. Put $s = a+b+c$ and let the original number be M. Since the other five sum to $3194,$

$$M = 222s - 3194.$$

Because $100 \le M \le 999,$ we need $15 \le s \le 18.$ Testing these four values gives $M = 136, 358, 580, 802,$ respectively. Only 358 has digit sum equal to its assumed value, namely $16.$ Therefore the original number is $358.$

11. The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form

$$a_0 + a_1 y + a_2 y^2 + \cdots \\ + a_{16} y^{16} + a_{17} y^{17},$$

where $y = x + 1$ and the a_i's are constants. Find a_2.

Solution:

The polynomial is $\sum_{k=0}^{17} (-x)^k$. Since $x = y - 1$, this becomes

$$\sum_{k=0}^{17} (1 - y)^k.$$

For $k \geq 2$, the coefficient of y^2 in $(1 - y)^k$ is $\binom{k}{2}$. Hence

$$a_2 = \sum_{k=2}^{17} \binom{k}{2} = \binom{18}{3} = 816.$$

12. Let the sum of a set of numbers be the sum of its elements. Let S be a set of positive integers, none greater than 15. Suppose no two disjoint subsets of S have the same sum. What is the largest sum a set S with these properties can have?

Solution:

First, S has at most five elements. If it had six elements $s_1, \ldots, s_6$, then its 64 subset sums would all be distinct: equality between two subset sums, after cancelling their common elements, would violate the given condition.

Choose a subset uniformly at random and let X be its sum. Then

$$\begin{aligned} \mathrm{Var}(X) &= \frac{1}{4}\sum_{i=1}^{6} s_i^2 \\ &\leq \frac{1}{4}(10^2 + 11^2 + \cdots + 15^2) \\ &= \frac{955}{4}. \end{aligned}$$

On the other hand, X is uniform on 64 distinct integers. The least possible variance for 64 distinct integers occurs when they are consecutive, and is

$$\frac{64^2 - 1}{12} = \frac{1365}{4},$$

a contradiction.

A set with at most four elements has sum at most $12 + 13 + 14 + 15 = 54$. There are only seven five-element subsets of $\{1, \ldots, 15\}$ whose sums are at least 62. Each fails, as witnessed by the following equal sums:

$\{8, 12, 13, 14, 15\} : 13 + 14 = 12 + 15$. $\{9, 11, 13, 14, 15\} : 11 + 13 = 9 + 15$. $\{9, 12, 13, 14, 15\} : 13 + 14 = 12 + 15$.

$\{10, 11, 12, 14, 15\} : 11 + 14 = 10 + 15$. $\{10, 11, 13, 14, 15\} : 11 + 13 = 10 + 14$.

$\{10, 12, 13, 14, 15\} : 12 + 13 = 10 + 15$. $\{11, 12, 13, 14, 15\} : 12 + 13 = 11 + 14$.

Thus the answer is at most 61.

The set $\{8, 11, 13, 14, 15\}$ attains 61. Its 32 subset sums, in order, are

$$\begin{gathered} 0, 8, 11, 13, 14, 15, 19, 21, \\ 22, 23, 24, 25, 26, 27, 28, 29, \\ 32, 33, 34, 35, 36, 37, 38, 39, \\ 40, 42, 46, 47, 48, 50, 53, 61, \end{gathered}$$

all distinct. Equal sums from arbitrary subsets would, after deleting their intersection, give equal sums from disjoint subsets, so this verifies the required property.

13. In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and so on. We denote these by TH, HH, and so on. For example, in the sequence HHTTHHHHTHHTTTT of 15 coin tosses, there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses contain exactly two HH, three HT, four TH, and five TT subsequences?

Solution:

Since there are four TH transitions and three HT transitions, every valid sequence starts with T and ends with H. It therefore has four T-runs and four H-runs, alternating.

If the H-runs have total length h, then the number of HH transitions is $h - 4$. Thus $h = 6$, and the positive lengths of the four H-runs can be chosen in $\binom{6-1}{4-1} = \binom{5}{3} = 10$ ways. Similarly, five TT transitions mean that the four T-runs have total length 9, giving $\binom{9-1}{4-1} = \binom{8}{3} = 56$ choices. The alternating order is fixed, so the number of sequences is $10 \cdot 56 = 560$.

14. The shortest distances between an interior diagonal of a rectangular parallelepiped P and the edges it does not meet are $2\sqrt{5}$, $\frac{30}{\sqrt{13}}$, and $\frac{15}{\sqrt{10}}$. Determine the volume of P.

Solution:

Let the side lengths be a, b, c. For example, the distance from the space diagonal with direction (a, b, c) to a nonintersecting edge parallel to the a-direction is

$$\frac{bc}{\sqrt{b^2+c^2}}.$$

The other two distances are obtained cyclically. We may assign the three given distances to these three directions in the listed order, since permuting them only permutes the side lengths.

Put $X = \frac{1}{a^2}$, $Y = \frac{1}{b^2}$, and $Z = \frac{1}{c^2}$. Taking reciprocal squares gives

$$\begin{aligned} Y + Z &= \frac{1}{20}, \\ X + Z &= \frac{13}{900}, \\ X + Y &= \frac{2}{45}. \end{aligned}$$

Solving,

$$\begin{aligned} X &= \frac{1}{225}, \\ Y &= \frac{1}{25}, \\ Z &= \frac{1}{100}. \end{aligned}$$

Hence the side lengths are $15, 5, 10$, and the volume is $15 \cdot 5 \cdot 10 = 750$.

15. Let triangle ABC be a right triangle in the xy-plane with a right angle at C. Given that the hypotenuse AB has length 60, and that the medians through A and B lie along the lines $y = x + 3$ and $y = 2x + 4$, respectively, find the area of $\triangle ABC$.

Solution:

The median lines meet at the centroid $G = (-1, 2)$. For real u, v, write

$$\begin{aligned} A &= G + (u, u), \\ B &= G + (v, 2v). \end{aligned}$$

Since $A + B + C = 3G$,

$$C = G - (u, u) - (v, 2v).$$

The condition $AC \perp BC$ gives

$$\begin{aligned} &(2u + v, 2u + 2v) \\ &\qquad \cdot (u + 2v, u + 4v) = 0, \end{aligned}$$

or

$$4u^2 + 15uv + 10v^2 = 0.$$

Meanwhile $AB = 60$ gives

$$\begin{aligned} &(u - v)^2 + (u - 2v)^2 \\ &\qquad = 2u^2 - 6uv + 5v^2 \\ &\qquad = 3600. \end{aligned}$$

Subtracting half of the first equation from the second yields $-\frac{27}{2}uv = 3600$, so $uv = -\frac{800}{3}$.

Using the two perpendicular legs from C, the area is half the absolute determinant:

$$\begin{aligned}[ABC] &= \frac{1}{2}\left|\det\begin{pmatrix}2u+v & 2u+2v\\ u+2v & u+4v\end{pmatrix}\right|\\ &= \frac{1}{2}|3uv| = 400.\end{aligned}$$

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