1986 AIME

Time limit: 180 minutes

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1. What is the sum of the solutions to the equation $\sqrt[4]{x} = \frac{12}{7-\sqrt[4]{x}}$?

2. Evaluate the product

$$\begin{gathered}(\sqrt{5}+\sqrt{6}+\sqrt{7}) \\ \cdot(-\sqrt{5}+\sqrt{6}+\sqrt{7}) \\ \cdot(\sqrt{5}-\sqrt{6}+\sqrt{7}) \\ \cdot(\sqrt{5}+\sqrt{6}-\sqrt{7}).\end{gathered}$$

3. If $\tan x+\tan y=25$ and $\cot x+\cot y=30,$ what is $\tan(x+y)$?

4. Determine $3x_4 + 2x_5$ if $x_1, x_2, x_3, x_4,$ and x_5 satisfy the system

$$\begin{aligned}
2x_1 + x_2 + x_3 + x_4 + x_5 &= 6, \\
x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12, \\
x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24, \\
x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48, \\
x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96.
\end{aligned}$$

5. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

6. The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in an incorrect sum of 1986. What was the number of the page that was added twice?

7. The increasing sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term of this sequence.

8. Let S be the sum of the base 10 logarithms of all the proper divisors of 1000000. What is the integer nearest to S?

9. In $\triangle ABC$, $AB = 425$, $BC = 450$, and $AC = 510$. An interior point P is drawn, and segments are drawn through P parallel to the sides of the triangle. If these three segments have equal length d, find d.

10. In a parlor game, the magician asks one of the participants to think of a three-digit number (abc), where a, b, and c represent base-10 digits in the indicated order. The magician then asks this person to form the numbers (acb), (bca), (bac), (cab), and (cba), to add these five numbers, and to reveal their sum N. If told N, the magician can identify the original number (abc). Play the role of the magician and determine (abc) if $N = 3194$.

11. The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form

$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17},$$

where $y = x + 1$ and the a_i's are constants. Find a_2.

12. Let the sum of a set of numbers be the sum of its elements. Let S be a set of positive integers, none greater than $15.$ Suppose no two disjoint subsets of S have the same sum. What is the largest sum a set S with these properties can have?

13. In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and so on. We denote these by TH, HH, and so on. For example, in the sequence HHTTHHHHTHHTTTT of 15 coin tosses, there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses contain exactly two HH, three HT, four TH, and five TT subsequences?

14. The shortest distances between an interior diagonal of a rectangular parallelepiped P and the edges it does not meet are $2\sqrt{5},$ $\frac{30}{\sqrt{13}}$, and $\frac{15}{\sqrt{10}}.$ Determine the volume of $P.$

15. Let triangle ABC be a right triangle in the xy-plane with a right angle at $C.$ Given that the hypotenuse AB has length $60,$ and that the medians through A and B lie along the lines $y = x + 3$ and $y = 2x + 4,$ respectively, find the area of $\triangle ABC.$

Solutions: https://live.poshenloh.com/past-contests/aime/1986/solutions