1985 AIME Solutions

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1. Let $x_1 = 97$, and for $n > 1$ let $x_n = \frac{n}{x_{n-1}}$. Calculate the product $x_1 x_2 \cdots x_8$.

Solution:

The recurrence gives $x_{n-1} x_n = n$. Therefore

$$\begin{aligned} x_1 x_2 \cdots x_8 &= (x_1 x_2)(x_3 x_4) \\ &\quad \cdot (x_5 x_6)(x_7 x_8) \\ &= 2 \cdot 4 \cdot 6 \cdot 8 \\ &= 384. \end{aligned}$$

2. When a right triangle is rotated about one leg, the volume of the cone produced is $800\pi\ \mathrm{cm}^3$. When the triangle is rotated about the other leg, the volume of the cone produced is $1920\pi\ \mathrm{cm}^3$. What is the length (in cm) of the hypotenuse of the triangle?

Solution:

Let the legs be a and b. In a suitable order,

$$\frac{1}{3}\pi b^2 a = 800\pi,$$
$$\frac{1}{3}\pi a^2 b = 1920\pi.$$

Their ratio gives $\frac{a}{b} = \frac{12}{5}$, so write $a = 12k$ and $b = 5k$. The first equation becomes $100k^3 = 800$, hence $k = 2$. The legs are 24 and 10, so the hypotenuse is $\sqrt{24^2 + 10^2} = 26$.

3. Find c if $a, b,$ and c are positive integers which satisfy $c = (a + bi)^3 - 107i$, where $i^2 = -1$.

Solution:

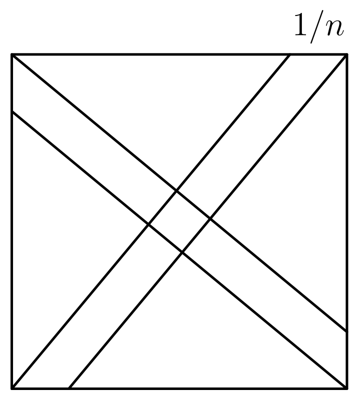
Expanding and using that c is real gives

$$b(3a^2 - b^2) = 107.$$

Since 107 is prime, $b = 1$ or 107. The latter would require $3a^2 = 11450$, which is impossible. Thus $b = 1$, and $3a^2 - 1 = 107$, so $a = 6$. The real part is

$$c = a^3 - 3ab^2 = 216 - 18 = 198.$$

4. A small square is constructed inside a square of area 1 by dividing each side of the unit square into n equal parts, and then connecting the vertices to the division points closest to the opposite vertices. Find the value of n if the area of the small square is exactly $\frac{1}{1985}$.



Solution:

Put the outer square at $(0,0),(1,0),(1,1),(0,1)$. One pair of construction lines has equations

$$\begin{aligned} nx-(n-1)y&=0, \\ nx-(n-1)y&=1. \end{aligned}$$

The other pair is perpendicular to this pair, and the two pairs have the same separation. Thus the inner square has side length

$$\frac{1}{\sqrt{n^2+(n-1)^2}}$$

and area $\frac{1}{n^2+(n-1)^2}$. Hence

$$n^2+(n-1)^2=1985,$$

or $n^2-n-992=0$. Its positive root is $n=\frac{1+63}{2}=32$.

5. A sequence of integers $a_1, a_2, a_3, \ldots$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of the first 2001 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492?

Solution:

Put $a_1 = x$ and $a_2 = y$. The first six terms are

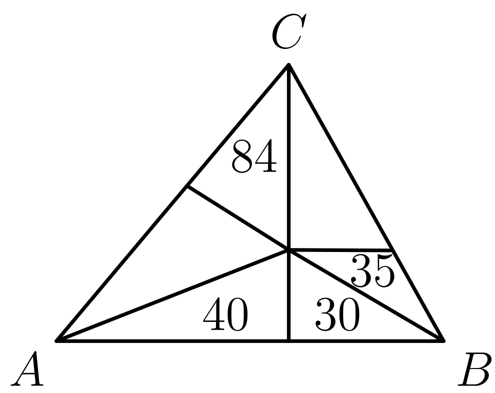
$$x,\ y,\ y-x,\ -x,\ -y,\ x-y,$$

after which the sequence repeats; these six terms sum to 0. Since $1492 \equiv 4 \pmod 6$ and $1985 \equiv 5 \pmod 6$, the given equations are

$$\begin{aligned} 2y - x &= 1985, \\ y - x &= 1492. \end{aligned}$$

Thus $y = 493$. Since $2001 \equiv 3 \pmod 6$, the requested sum is $x + y + (y - x) = 2y = 986$.

6. As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



Solution:

Let the unlabeled upper-right and upper-left areas be x and y, respectively. The three side-division ratios and Ceva's theorem give

$$\frac{4}{3} \cdot \frac{35}{x} \cdot \frac{84}{y} = 1,$$

so $xy = 3920.$

The cevian from A meets BC. The ratio of the two segments of BC is $\frac{35}{x}$. Computing the same ratio from the two large triangles with vertex A gives

$$\frac{35}{x} = \frac{40+30+35}{x+y+84},$$

so $y + 84 = 2x$. Solving with $xy = 3920$ yields $x = 70$ and $y = 56$. Therefore

$$\begin{aligned}[ABC] &= 84 + 70 + 35 \\ &\quad + 30 + 40 + 56 = 315.\end{aligned}$$

7. Assume that $a, b, c,$ and d are positive integers such that $a^5 = b^4, c^3 = d^2$, and $c - a = 19$. Determine $d - b$.

Solution:

Comparing prime exponents, write

$$\begin{aligned} a = t^4, \quad b = t^5, \\ c = s^2, \quad d = s^3 \end{aligned}$$

for positive integers s, t. Then

$$(s - t^2)(s + t^2) = s^2 - t^4 = 19.$$

Since 19 is prime, the factors are 1 and 19, giving $s = 10$ and $t^2 = 9$. Thus $t = 3$, and

$$\begin{aligned} d - b &= 10^3 - 3^5 \\ &= 1000 - 243 = 757. \end{aligned}$$

8. The sum of the following seven numbers is exactly 19 :

$$a_1 = 2.56,$$
$$a_2 = 2.61,$$
$$a_3 = 2.65,$$
$$a_4 = 2.71,$$
$$a_5 = 2.79,$$
$$a_6 = 2.82,$$
$$a_7 = 2.86.$$

It is desired to replace each a_i by an integer approximation $A_i, 1 \leq i \leq 7,$ so that the sum of the A_i's is also 19 and so that M, the maximum of the errors $|A_i - a_i|$, is as small as possible. For this minimum M, what is $100M$?

Solution:

Choose $A_1 = A_2 = 2$ and $A_3 = \cdots = A_7 = 3$. The sum is 19, and the largest error is $|A_2 - a_2| = 0.61$.

If $M < 0.61$, then $A_2, \ldots, A_7$ must all equal 3, and A_1 can only be 2 or 3. Their sum would therefore be at least 20, a contradiction. Thus the minimum is $M = 0.61$, and $100M = 61$.

9. In a circle, parallel chords of lengths $2, 3,$ and 4 determine central angles of $\alpha, \beta,$ and $\alpha+\beta$ radians, respectively, where $\alpha+\beta<\pi$. If $\cos\alpha$, which is a positive rational number, is expressed as a fraction in lowest terms, what is the sum of its numerator and denominator?

Solution:

Put $k=\frac{1}{2R}$, $x=\cos(\frac{\alpha}{2})$, and $y=\cos(\frac{\beta}{2})$. The chord data give

$$\sin\frac{\alpha}{2}=2k,$$
$$\sin\frac{\beta}{2}=3k,$$
$$\sin\frac{\alpha+\beta}{2}=4k.$$

The addition formula yields $2y+3x=4$. Also

$$\frac{1-x^2}{4}=\frac{1-y^2}{9},$$

so $9x^2-4y^2=5$. Substituting $y=\frac{4-3x}{2}$ gives $x=\frac{7}{8}$. Hence

$$\cos\alpha=2x^2-1=\frac{17}{32},$$

and the requested sum is $17+32=49$.

10. How many of the first 1000 positive integers can be expressed in the form

$$\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor,$$

where x is a real number, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z?

Solution:

Let $y = 2x = m + r$, where m is an integer and $0 \leq r < 1$. The expression is

$$10m + \lfloor r \rfloor + \lfloor 2r \rfloor + \lfloor 3r \rfloor + \lfloor 4r \rfloor.$$

Checking the breakpoints $r = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ shows that the fractional-part contribution takes exactly the values $0, 1, 2, 4, 5, 6$. Thus precisely six residue classes modulo 10 are attainable. Among the first 1000 positive integers, this gives $100 \cdot 6 = 600$.

11. An ellipse has foci at $(9, 20)$ and $(49, 55)$ in the xy-plane and is tangent to the x-axis. What is the length of its major axis?

Solution:

Reflect $(49, 55)$ across the x-axis to $(49, -55)$. For a point P on the x-axis, the sum of its distances to the original foci equals the length of a broken path from $(9, 20)$ through P to $(49, -55)$. Its minimum is the straight-line distance

$$\begin{aligned} &\sqrt{(49-9)^2 + (-55-20)^2} \\ &= \sqrt{40^2 + 75^2} = 85. \end{aligned}$$

Tangency means that the ellipse's constant distance sum equals this minimum. That sum is the major-axis length, so the answer is 85.

12. Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = \frac{n}{729}$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n.

Solution:

Let p_k be the probability that the bug is at A after k steps. It cannot stay at A, while from any other vertex it moves to A with probability $\frac{1}{3}$. Hence

$$p_{k+1} = \frac{1 - p_k}{3}, \qquad p_0 = 1.$$

Solving this recurrence gives

$$p_k = \frac{1}{4} + \frac{3}{4}\left(-\frac{1}{3}\right)^k.$$

Thus $p_7 = \frac{1}{4} - \frac{3}{4 \cdot 3^7} = \frac{182}{729}$, so $n = 182$.

13. The numbers in the sequence $101, 104, 109, 116, \ldots$ are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

Solution:

A common divisor d_n divides

$$a_{n+1} - a_n = 2n + 1.$$

It therefore also divides

$$\begin{aligned} 4(n^2 + 100) - (2n + 1)^2 \\ + 2(2n + 1) = 401. \end{aligned}$$

Since 401 is prime, $d_n \leq 401$. Equality occurs at $n = 200$, because $2n + 1 = 401$ and $n^2 + 100 = 40100 = 100 \cdot 401$. Hence the maximum is 401.

14. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned $\frac{1}{2}$ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned against the ten players with the least number of points. (In particular, each of the ten lowest-scoring players earned half of her or his points against the other nine of the ten.) What was the total number of players in the tournament?

Solution:

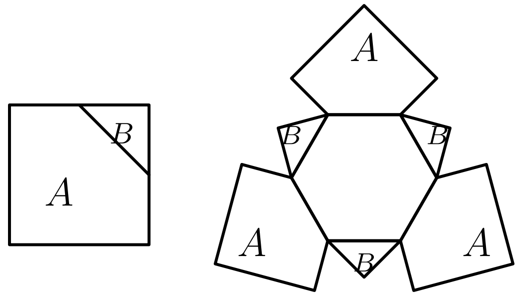
The games among the ten lowest players contribute $\binom{10}{2} = 45$ total points. These are half of those ten players' combined score, so their combined score is $90.$ Hence they earned 45 points in games against the other m players.

The other players therefore earned $10m - 45$ points against the lowest ten. By the condition, this is half their combined score, which is $\binom{m+10}{2} - 90.$ Thus

$$2(10m - 45) = \binom{m+10}{2} - 90,$$

giving $(m-6)(m-15) = 0.$ If $m = 6,$ the lowest ten average 9 points while the other six average only $5,$ impossible for the latter group to rank above them. Hence $m = 15,$ and the total number of players is $10 + 15 = 25.$

15. Three $12\ \mathrm{cm} \times 12\ \mathrm{cm}$ squares are each cut into two pieces A and B, as shown in the first figure below, by joining the midpoints of two adjacent sides. These six pieces are then attached to a regular hexagon, as shown in the second figure, so as to fold into a polyhedron. What is the volume (in cm^3) of this polyhedron?



Solution:

Consider a $12 \times 12 \times 12$ cube and the plane through the midpoints of the six edges that join opposite groups of three vertices. In coordinates $0 \leq x, y, z \leq 12$, this is the plane $x + y + z = 18$. Its cross-section is a regular hexagon with side $6\sqrt{2}$, the same as the cut edge joining adjacent side midpoints.

On three faces of the cube, the plane leaves a square with a corner triangle removed, exactly piece A. On the other three faces, it leaves the complementary right-isosceles corner triangle, exactly piece B. Thus the pictured net is one of the two pieces into which this plane cuts the cube. Central symmetry interchanges the two pieces, so each has half the cube's volume:

$$V = \frac{12^3}{2} = 864.$$

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