1984 AIME Solutions

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1. Find the value of $a_2 + a_4 + a_6 + \cdots + a_{98}$ if $a_1, a_2, a_3, \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \cdots + a_{98} = 137$.

Solution:

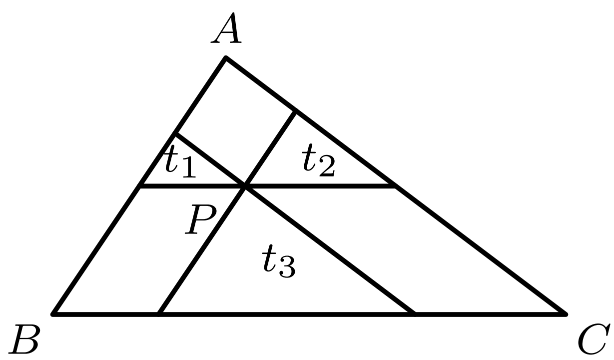
Let $E = a_2 + a_4 + \cdots + a_{98}$ and $O = a_1 + a_3 + \cdots + a_{97}$. There are 49 pairs, and $a_{2k} - a_{2k-1} = 1$, so $E - O = 49$. Also $E + O = 137$. Adding the two equations gives $2E = 186$, and hence $E = 93$.

2. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 8. Compute $\frac{n}{15}$.

Solution:

A multiple of 15 must end in 0 and have digit sum divisible by 3. Because $8 \equiv 2 \pmod{3}$, the number of 8's must be a positive multiple of 3. The smallest possible number therefore has three 8's followed by 0, namely $n = 8880$. Thus $\frac{n}{15} = 592$.

3. A point P is chosen in the interior of $\triangle ABC$ so that when lines are drawn through P parallel to the sides of $\triangle ABC$, the resulting smaller triangles, t_1, t_2, and t_3 in the figure, have areas $4, 9$, and 49, respectively. Find the area of $\triangle ABC$.



Solution:

Let the area of $\triangle ABC$ be K. The three small triangles are similar to $\triangle ABC$, so their corresponding linear ratios are

$$\frac{2}{\sqrt{K}}, \qquad \frac{3}{\sqrt{K}}, \qquad \frac{7}{\sqrt{K}}.$$

Each such scale factor is also the perpendicular distance from P to one side divided by the altitude to that side. These are the three barycentric area ratios $\frac{[PBC]}{K}$, $\frac{[PCA]}{K}$, and $\frac{[PAB]}{K}$, which add to 1. Hence

$$\frac{2+3+7}{\sqrt{K}} = 1.$$

Therefore $\sqrt{K} = 12$ and $K = 144$.

4. Let S be a list of positive integers—not necessarily distinct—in which the number 68 appears. The average (arithmetic mean) of the numbers in S is 56. However, if 68 is removed, the average of the remaining numbers drops to 55. What is the largest number that can appear in S?

Solution:

If the original list has N entries, then

$$56N - 68 = 55(N-1),$$

so $N = 13$ and the total of the entries is $56 \cdot 13 = 728$. Besides the entry 68, make eleven entries equal to the least positive integer, 1. The remaining entry is then

$$728 - 68 - 11 = 649.$$

This construction is valid, and no larger entry is possible.

5. Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

Solution:

Let $A = \log_2 a$ and $B = \log_2 b$. The two equations become

$$\frac{A}{3} + B = 5, \qquad \frac{B}{3} + A = 7.$$

Equivalently, $A + 3B = 15$ and $3A + B = 21$, which give $A = 6$ and $B = 3$. Therefore $ab = 2^{A+B} = 2^9 = 512$.

6. Three circles, each of radius 3, are drawn with centers at $(14, 92)$, $(17, 76)$, and $(19, 84)$. A line passing through $(17, 76)$ is such that the total area of the parts of the three circles to one side of the line is equal to the total area of the parts of the three circles to the other side of it. What is the absolute value of the slope of this line?

Solution:

For a circle of radius 3, let $G(d)$ be the signed difference between the areas on the two sides of a line when the center's signed distance from the line is d. The function G is odd and strictly increasing for $-3 < d < 3$, and is constant only after the line no longer cuts the circle. The circle centered at $(17, 76)$ contributes zero, so the line must separate the other two centers.

Let $M = (\frac{33}{2}, 88)$ be the midpoint of those two centers. Among the lines through $(17, 76)$ that separate them, the smaller of their two distances to the line is largest when the distances are equal, namely for the line through M. Even then, the distance is $\frac{56}{\sqrt{577}} < 3$, so the line must cut at least one of the two circles. Balance then forces it to cut both. The strict increase of G therefore forces the two signed distances to be opposites, so the required line passes through M. Its slope is

$$\frac{88 - 76}{\frac{33}{2} - 17} = -24.$$

Its absolute value is 24.

7. The function f is defined on the set of integers and satisfies $f(n) = n - 3$ if $n \geq 1000$, and

$$f(n) = f(f(n+5))$$

if $n < 1000$. Find $f(84)$.

Solution:

Directly from the definition,

$$\begin{aligned} f(999) &= f(f(1004)) \\ &= f(1001) = 998, \\ f(998) &= f(f(1003)) \\ &= f(1000) = 997. \end{aligned}$$

Continuing gives $f(997) = 998$, $f(996) = 997$, and $f(995) = 998$.

We now use downward induction. If $n < 995$ is even, then $n + 5$ is odd, so the established pattern above n gives $f(n) = f(f(n+5))$. Thus $f(n) = f(998) = 997$. If n is odd, the same argument gives $f(n) = f(997) = 998$. Thus every even $n < 1000$ has value 997. Since 84 is even, $f(84) = 997$.

8. The equation $z^6 + z^3 + 1 = 0$ has one complex root with argument θ between 90° and 180° in the complex plane. Determine the degree measure of θ.

Solution:

Let $u = z^3$. Then $u^2 + u + 1 = 0$, so u has argument 120° or 240°. Taking cube roots gives possible arguments

$$40^\circ, 160^\circ, 280^\circ, 80^\circ, 200^\circ, 320^\circ.$$

The only one strictly between 90° and 180° is 160°.

9. In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is $15\ \mathrm{cm}^2$ and the area of face ABD is $12\ \mathrm{cm}^2$. These two faces meet each other at a 30° angle. Find the volume of the tetrahedron in cm^3.

Solution:

Let h_C and h_D be the perpendicular distances from C and D to AB. From the two face areas,

$$h_C = \frac{2 \cdot 15}{3} = 10,$$
$$h_D = \frac{2 \cdot 12}{3} = 8.$$

The angle between these two perpendicular directions is the 30° dihedral angle. Hence the scalar triple product gives

$$\begin{aligned} V &= \frac{1}{6}(AB)h_C h_D \sin 30^\circ \\ &= \frac{1}{6} \cdot 3 \cdot 10 \cdot 8 \cdot \frac{1}{2} \\ &= 20. \end{aligned}$$

10. Mary told John her score on the American High School Mathematics Examination (AHSME), which was over 80. From this, John was able to determine the number of problems Mary solved correctly. If Mary's score had been any lower, but still over 80, John could not have determined this. What was Mary's score? (Recall that the AHSME consists of 30 multiple-choice problems and that one's score, s, is computed by the formula $s = 30 + 4c - w$, where c is the number correct and w is the number wrong; students are not penalized for problems left unanswered.)

Solution:

From $s = 30 + 4c - w$, we have $w = 30 + 4c - s$. The conditions $w \geq 0$ and $30 - c - w \geq 0$ give

$$\left\lceil \frac{s-30}{4} \right\rceil \leq c \leq \left\lfloor \frac{s}{5} \right\rfloor .$$

Evaluating these integer endpoints for scores 81 through 118 always leaves at least two possible values of c. At $s = 119$, both endpoints equal 23, so John can determine $c = 23$. Thus the first score over 80 with the required property is 119.

11. A gardener plants three maple trees, four oak trees, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{m}{n}$ in lowest terms be the probability that no two birch trees are next to one another. Find $m + n$.

Solution:

The five birch positions form a uniformly chosen 5-element subset of the 12 positions, so there are $\binom{12}{5}$ possibilities. After the seven non-birch trees are placed, there are eight gaps, including the two end gaps. Choosing five distinct gaps gives $\binom{8}{5}$ arrangements with no adjacent birches. Therefore

$$\frac{m}{n} = \frac{\binom{8}{5}}{\binom{12}{5}} = \frac{56}{792} = \frac{7}{99},$$

and $m + n = 106$.

12. A function f is defined for all real numbers and satisfies

$$f(2+x) = f(2-x),$$
$$f(7+x) = f(7-x)$$

for all real x. If $x = 0$ is a root of $f(x) = 0$, what is the least number of roots $f(x) = 0$ must have in the interval $-1000 \leq x \leq 1000$?

Solution:

The identities make the root set invariant under reflection about 2 and about 7. Their composition is translation by 10. Starting from the root 0, these operations force every number in

$$\{10k : k \in \mathbb{Z}\}$$
$$\cup \{4 + 10k : k \in \mathbb{Z}\}$$

to be a root. In the interval, the first set contributes 201 roots and the second contributes 200, for a total of 401.

This bound is attainable: define f to be 0 on this invariant set and 1 elsewhere. It has both required reflection symmetries. Hence the least possible number is 401.

13. Find the value of

$$10\cot\left(\cot^{-1}3+\cot^{-1}7 \\ +\cot^{-1}13+\cot^{-1}21\right).$$

Solution:

Let $\alpha=\cot^{-1}3+\cot^{-1}7$. The cotangent addition formula gives

$$\begin{aligned}\cot\alpha &= \frac{3\cdot 7-1}{3+7} \\ &= 2.\end{aligned}$$

Combining the next angle gives

$$\frac{2\cdot 13-1}{2+13}=\frac{5}{3},$$

and combining the last gives

$$\frac{\left(\frac{5}{3}\right)\cdot 21-1}{\frac{5}{3}+21}=\frac{3}{2}.$$

Multiplying by 10 yields 15.

14. What is the largest even integer that cannot be written as the sum of two odd composite numbers? (Recall that a positive integer is said to be composite if it is divisible by at least one positive integer other than 1 and itself.)

Solution:

In a representation of 38, the smaller odd composite would be at most 19. The only possibilities are 9 and 15, whose complements 29 and 23 are prime. Thus 38 is not representable.

Now let $N > 38$ be even. If $N \equiv 0 \pmod 6$, write $N = 9 + (N - 9)$. If $N \equiv 2 \pmod 6$, write $N = 35 + (N - 35)$. If $N \equiv 4 \pmod 6$, write $N = 25 + (N - 25)$. In each case the second summand is an odd multiple of 3 greater than 3, hence is composite; the fixed first summand is also odd and composite. Therefore every even integer greater than 38 is representable, so the largest exception is 38.

15. Determine $w^2 + x^2 + y^2 + z^2$ if

$$\frac{x^2}{2^2-1} + \frac{y^2}{2^2-3^2} + \frac{z^2}{2^2-5^2} + \frac{w^2}{2^2-7^2} = 1,$$
$$\frac{x^2}{4^2-1} + \frac{y^2}{4^2-3^2} + \frac{z^2}{4^2-5^2} + \frac{w^2}{4^2-7^2} = 1,$$
$$\frac{x^2}{6^2-1} + \frac{y^2}{6^2-3^2} + \frac{z^2}{6^2-5^2} + \frac{w^2}{6^2-7^2} = 1,$$
$$\frac{x^2}{8^2-1} + \frac{y^2}{8^2-3^2} + \frac{z^2}{8^2-5^2} + \frac{w^2}{8^2-7^2} = 1.$$

Solution:

Define

$$R(T) = \frac{x^2}{T-1} + \frac{y^2}{T-9} + \frac{z^2}{T-25} + \frac{w^2}{T-49} - 1.$$

Put

$$\begin{aligned} N(T) &= (T-4)(T-16) \\ &\quad \cdot (T-36)(T-64), \\ D(T) &= (T-1)(T-9) \\ &\quad \cdot (T-25)(T-49). \end{aligned}$$

With common denominator $D(T)$, the numerator of $R(T)$ has leading coefficient -1. The four equations say its zeros are $4, 16, 36,$ and 64. Therefore $R(T) = -\frac{N(T)}{D(T)}$.

Let $S = w^2 + x^2 + y^2 + z^2$. As T tends to infinity, the defining expression gives

$$R(T) = -1 + \frac{S}{T} + O(T^{-2}).$$

On the other hand, the sum of the four numerator roots is 120, while the sum of the four denominator roots is 84, so the factored expression gives

$$\begin{aligned} R(T) = -1 + \frac{120-84}{T} \\ + O(T^{-2}). \end{aligned}$$

Hence $w^2 + x^2 + y^2 + z^2 = 36$.

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